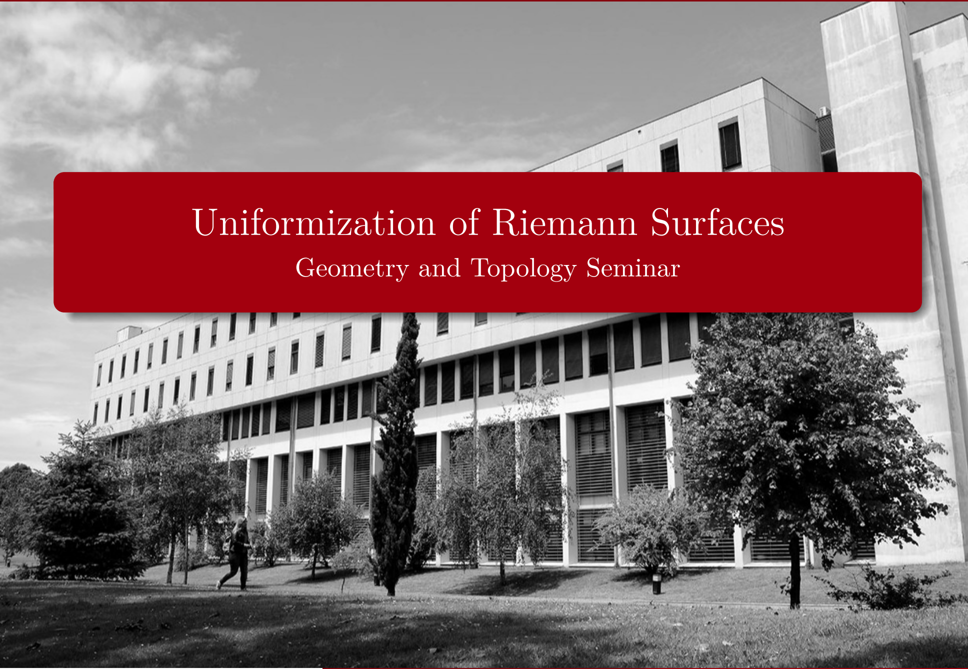

Uniformization of Riemann Surfaces

Geometry and Topology Seminar

Main References

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(2) B. Rodin & D. Sullivan, The convergence of circle packings to the Riemann mapping,
J. Differential Geom. 26 (1987), no. 2, 349–360.

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Riemann Surfaces

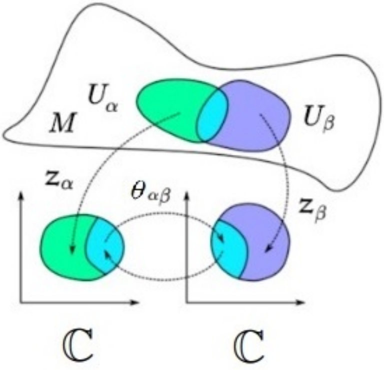
Let M be a connected, differentiable 2-manifold (a connected surface). We say that M is a Riemann surface if there exists a complex atlas $\{(U_\alpha, z_\alpha)\}$ of M, i.e. an atlas $\{(U_\alpha, z_\alpha)\}$ satisfying

- $(U_\alpha)_\alpha$ is an open cover of M;
- $z_\alpha : U_\alpha \longrightarrow \mathbb{C}$ is a homeomorphism onto (an open set of) $\mathbb{C}$;
- the change of coordinates

$$\theta_{\alpha\beta} = z_\alpha \circ {z_\beta}^{-1} : z_\beta(U_\alpha \cap U_\beta) \longrightarrow z_\alpha(U_\alpha \cap U_\beta)$$

  are holomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$.

▶ The mappings z_α are called (complex) local coordinates.


M
U_α
U_β
z_α
$\theta_{\alpha\beta}$
z_β
$\mathbb{C}$
$\mathbb{C}$

Holomorphic and conformal functions

Let M, N be two Riemann surfaces.

A map $\mathrm{f} : \mathrm{M} \longrightarrow \mathrm{N}$ is said holomorphic if for any local coordinates ϕ_α and ψ_β for M and N, respectively, that we fix, we have that $\phi_\alpha \circ \mathrm{f} \circ \psi_\beta^{-1}$ is holomorphic.

A map $\mathrm{f} : \mathrm{M} \longrightarrow \mathrm{N}$ is called conformal/bi-holomorphic if it is bijective and holomorphic.

Two Riemann surfaces M and N are said equivalent if there exists a conformal map from M to N. In this case, we write

$$\mathrm{M} \simeq \mathrm{N}.$$

Outline of The Riemann Uniformization Theorem

Theorem ($\mathfrak{Riemann\ Uniformization}$)

Let M be a simply connected Riemann surface. Then M is equivalent to one of the following three surfaces:

- $\mathbb{S}^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$;
- $\mathbb{C}$;
- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem ($\mathfrak{The\ Riemann\ Mapping\ Theorem}$)

If $M \subseteq \mathbb{C}$ is a bounded, simply connected and open domain, then:

$$M \simeq \mathbb{D}$$

Differential Forms

A 1-form ω on a Riemann surface M is a correspondence, for each local coordinate $z = x + i\,y$, of an ordered pair of functions (f, g) such that

$$\omega = f\,dx + g\,dy$$

is invariant under coordinate changes

Differential Forms

A 1-form ω on a Riemann surface M is a correspondence, for each local coordinate $z = x + i\,y$, of an ordered pair of functions (f, g) such that

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is invariant under coordinate changes, i.e. if $\tilde{z} = \tilde{x} + i\,\tilde{y}$ is another local coordinate whose domain of definition has a non-empty intersection with the one of z and if

$$\omega = \tilde{f}\,d\tilde{x} + \tilde{g}\,d\tilde{y}$$

is the correspondence in these coordinates, then (f, g) and $(\tilde{f}, \tilde{g})$ satisfy the following relation

Differential Forms

$$\begin{pmatrix} \tilde{f}(\tilde{z}) \\ \\ \tilde{g}(\tilde{z}) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \tilde{x}} & \frac{\partial y}{\partial \tilde{x}} \\ \\ \frac{\partial x}{\partial \tilde{y}} & \frac{\partial y}{\partial \tilde{y}} \end{pmatrix} \begin{pmatrix} f(z(\tilde{z})) \\ \\ g(z(\tilde{z})) \end{pmatrix} =$$

$$= \left(\mathfrak{Jac}(x, y)_{(\tilde{x},\tilde{y})} \right)^{T} \begin{pmatrix} f(z(\tilde{z})) \\ \\ g(z(\tilde{z})) \end{pmatrix}$$

Conjugation Operator

For 1-forms on a Riemann surface M, we define the conjugation operator $\star : \Lambda^1 M \to \Lambda^1 M$ as follows. If, in local coordinates $z = x + i\, y$,

$$\omega = f\, dx + g\, dy$$

then, in the same coordinates, we have

$$^\star\omega := -\, g\, dx + f\, dy\,.$$

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If ω is expressed in $z = x + iy$ and $\bar{z} = x - iy$ as

$$\omega = u(z, \bar{z})\,dz + v(z, \bar{z})\,d\bar{z}\,,$$

then

$$^\star\,\omega = -i\,u(z, \bar{z})\,dz + i\,v(z, \bar{z})\,d\bar{z}$$

Harmonic functions and differentials

If f is a function of class C^2 on a Riemann surface M, we define the Laplacian of f, in the local coordinates $z = x + iy$ as being the 2-form defined as

$$\underline{\Delta f} := (f_{xx} + f_{yy}) \, dx \wedge dy$$

It can be checked that these local expressions define a global 2-form. Alternatively, this also follows from noting that (locally)

$$\Delta f = d \star df .$$

A function $f : M \longrightarrow \mathbb{R}$ is said to be harmonic if $\Delta f \equiv 0$.

Harmonic functions and differentials [...]

A 1-form ω is harmonic if, locally, $\omega = \mathrm{d}f$ with f an harmonic function.

Theorem (Classification of harmonic differentials)

- ω is a harmonic 1-form iff ω is closed and co-closed.

Holomorphic Differentials

A 1-form ω is holomorphic if, locally, $\omega = \mathrm{d}f = \frac{\partial f}{\partial z}\,\mathrm{d}z$, with f holomorphic.

Theorem (Classification of holomorphic differentials)

- ω is holomorphic iff ω is closed and ${}^\star\omega = -i\omega$.

Super-harmonic functions

Remark: From now on, we will only consider real-valued functions on a Riemann surface M.

A continuous function $u : M \longrightarrow \mathbb{R}$ is called super-harmonic if for every given domain $D \subseteq M$ that we consider and for all harmonic function h satisfying $u \geq h$ on D, we have

$$u \equiv h \text{ on } D \qquad \text{or} \qquad u > h \text{ on } D.$$

Subharmonic functions can analogously be defined.

Perron Families of subharmonic functions

Let $\mathfrak{S}$ be a family of subharmonic functions on a Riemann surface M. The family $\mathfrak{S}$ is called a Perron family if and only if it satisfies the three conditions below.

I. $\mathfrak{S}$ is non−empty;

II. $\forall$conformal disc K in M, $\forall u \in \mathfrak{S}$, $\exists v \in \mathfrak{S}$: $v|_K$ is harmonic and $v \geq u$.

III. $\forall u_1, u_2 \in \mathfrak{S}$, $\exists v \in \mathfrak{S}$: $v \geq \max\{u_1, u_2\}$.

Theorem (Perron Principle)

Let $\mathfrak{S}$ be a Perron family on a Riemann surface M. If $s : M \to \mathbb{R}$ is defined as $s(P) := \sup_{u \in \mathfrak{S}}\{u(P)\}$, then either s is "identically $+\infty$" (by abuse of notation) or s is a harmonic function.

Riemann Existence Theorem

These techniques lead to the existence of harmonic functions with "poles" (i.e. harmonic functions defined on the complement of a finite set and having an infinite limit at these points).

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Every Riemann surface admits a non-constant meromorphic function.

Riemann Existence Theorem

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Theorem (Riemann Existence Theorem)

Every Riemann surface admits a non-constant meromorphic function.

Regardless of the Uniformization Theorem, the existence of non-constant meromorphic functions on every Riemann surface is a fundamental result that lies at the origin of other famous theorems (e.g. Riemann-Roch, the algebraic character of compact Riemann surfaces).

Elliptic, Parabolic and Hyperbolic Riemann surfaces

We may classify Riemann surfaces in three groups according to the existence (or non-existence) of special harmonic functions:

Elliptic: M is compact;

Parabolic: M is not compact and the only positive super-harmonic functions defined in M are the constants;

Hyperbolic: M possesses a non-constant positive super-harmonic function.

For simply connected Riemann surfaces

Theorem

If M is a simply connected hyperbolic Riemann surface, then:

$$\mathrm{M} \simeq \mathbb{D}$$

For simply connected Riemann surfaces

Theorem

If M is a simply connected hyperbolic Riemann surface, then:

$$M \simeq \mathbb{D}$$

If M is a simply connected elliptic (resp. parabolic) Riemann surface, then:

$$M \simeq \mathbb{S}^2 \text{ (resp. } \mathbb{C}\text{)}$$

Remark

The above classification does not respect quotients. This is particularly relevant in view of the following theorem:

Theorem

Every Riemann surface S is biholomorphically equivalent to D/Γ, where D is $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{S}^2$ and Γ is a group acting freely and properly discontinuously on D.

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Every Riemann surface S is biholomorphically equivalent to D/Γ, where D is $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{S}^2$ and Γ is a group acting freely and properly discontinuously on D.

A torus is a quotient of $\mathbb{C}$ and is an elliptic Riemann surface since it is compact. Nonetheless, its universal covering $\mathbb{C}$ is parabolic.

Revisiting the Riemann Mapping Theorem

Recall that:

Theorem

If $M \subseteq \mathbb{C}$ is a bounded, simply connected and open domain, then:

$$M \simeq \mathbb{D}$$

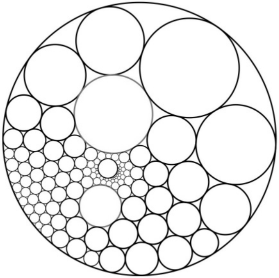
The proof of the Riemann Uniformization Theorem provides no information about the conformal maps above.

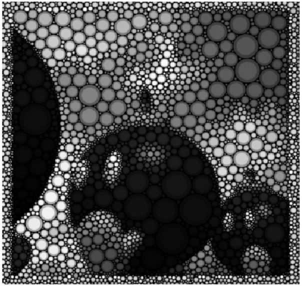
The main goal of the second part of the dissertation is to explore the approach of Rodin & Sullivan that provides an algorithm approximating the Riemann mappings. This algorithm gives us some geometric information on these maps and, to certain extent, can be used to investigate quantitative properties and boundary behavior.

Graphs and Circle Packings

A circle packing is a collection of circles $(C_\alpha)_{\alpha \in \Lambda}$ such that:

- if two circles, $C_1, C_2 \in (C_\alpha)_{\alpha \in \Lambda}$ intersect, this intersection must be tangential;
- $\bigcup_{\alpha \in \Lambda} C_\alpha$ forms a connected set.





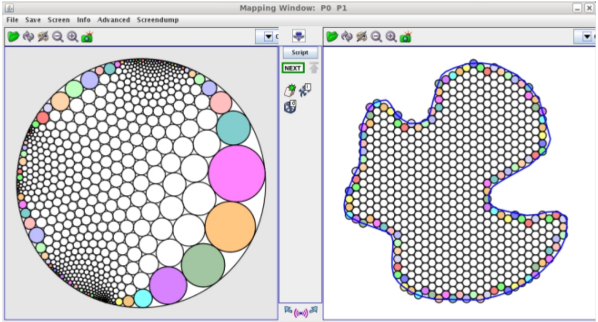
How to approximate Riemann mappings?

Let R be a bounded, open and simply connected region of $\mathbb{C}$.

Objective. To approximate the bi-holomorphic maps with domain R and image $\mathbb{D}$.

How?

- Fill in R with a sequence of smaller and more detailed regular hexagonal graphs.
- Invoke the Circle Packing Theorem in order to construct a bijection between the vertices on R and their images on $\mathbb{D}$.
- Consider the simplicial maps ϕ_{n} from the support of the graphs on R to the support of the image graphs. We only need to see that this sequence converges to a bijective, conformal mapping ϕ onto the unit disc.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Theorem: The sequence ϕ_n converges locally uniformly to a K$-$quasiconformal (for some $K > 0$) function ϕ which is, actually, 1-quasiconformal, i.e., conformal.

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Using Ackermann-Haïssinsky-Hinkkanen Theorem

- Quasi-conformality of ϕ:

Ring Lemma $\quad +$

Theorem 2. *Let U be a domain in the complex plane $\mathbb{C}$, and let $f : U \to f(U)$ be an orientation-preserving homeomorphism. Suppose that each $z \in U$ has a neighborhood W such that $\operatorname{Skew}(f|W)$ is finite.*

Suppose that $\sigma \geq 1$. If $\operatorname{skew}(f) \leq \sigma$ then f is $K(\sigma)$-quasiconformal where

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

In particular, if $\operatorname{skew}(f) = 1$ then f is a conformal mapping. The upper bound $K(\sigma)$ for the maximal dilatation of f is the best possible and is attained at least for certain affine mappings.

<u>Theorem:</u> The sequence ϕ_n converges locally uniformly to a K−quasiconformal (for some $K > 0$) function ϕ which is, actually, 1-quasiconformal, i.e., conformal.

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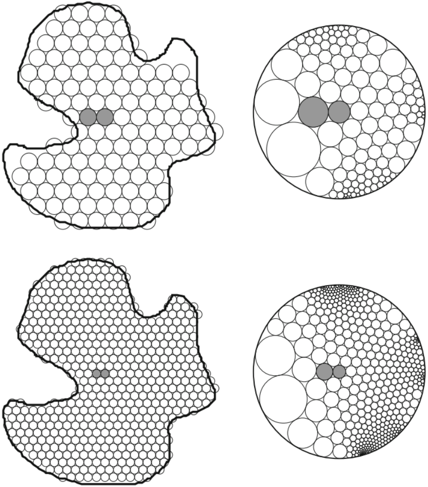
- ϕ is conformal:

Every ϕ_n is $(1 + s_n)$-quasiconformal, with $(s_n)_n \longrightarrow 0$ thanks to the Hexagonal Packing Lemma.



Going further on Circle Packings (or trying)

We have developed this theory of approximation by circle packings, however with a small caveat... this method only works for subsets of $\mathbb{C}$! The question remains:

Can we generalize it to any hyperbolic simply connected Riemann surface?

Let M be a hyperbolic simply connected Riemann surface inside $\mathbb{R}^3$.

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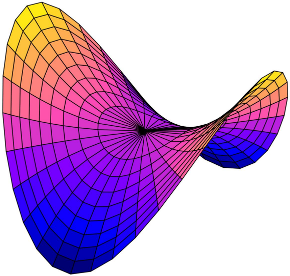

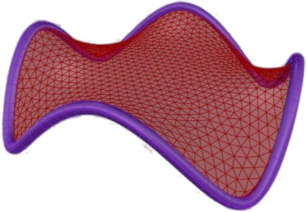

Equilateral Triangles

Different perspectives on constructing "equilateral triangles" on M:

Naive definition with local charts | Riemannian metric as the distance

Equilateral Triangles

Different perspectives on constructing ”equilateral triangles” on M:

Naive definition with local charts | Riemannian metric as the distance

Use the exterior Euclidian norm

Problem

How to send the vertices from M to points in $\mathbb{D}$? Still unanswered...
Continuing:

Define $\phi_n : |V_n| \longrightarrow "\mathbb{D}"$ as once again the simplicial maps originated by the vertices coming from the regular triangulation used on M.

- domains of ϕ_n are "approaching" M.
- ϕ_n are homeomorphisms.
- ϕ_n are quasi-conformal...

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- domains of ϕ_n are "approaching" M.
- ϕ_n are homeomorphisms.
- ϕ_n are quasi-conformal...
- ... and that's it (for now)!

Thank you !