Uniformization of Riemann Surfaces

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- Guilherme Nascimento

Resumo

O conceito de variedades diferenciáveis e, em particular, superfícies é essencial para o desenvolvimento da Geometria como conhecemos hoje. Neste sentido, temos usualmente em conta o espaço ambiente mais familiar: $\mathbb{R}^2$ e desenvolvemos uma teoria local à volta dos conhecimentos e propriedades adquiridos, provados e já estabelecidos.

No entanto, investigando mais a fundo, podemos averiguar que é possível introduzir uma estruturan complexa nestas superfícies, obtendo toda uma nova teoria adjacente a estes objetos: a teoria das Superfícies de Riemann.

Nesta dissertação, temos como próposito mergulhar neste conceito, suas propriedades, teoremas e generalizações de resultados familiares de Análise Complexa de modo a, no final da primeira parte, obter uma categorização total de todas as Superfícies de Riemann simplesmente conexas num enunciado simples e de enorme beleza presente no teorema da Uniformização de Riemann.

No final, também iremos abordar uma técnica de aproximação das funções bijetivas e holomorfas fornecidas pelo teorema da Uniformização no caso em que o domínio é um aberto simplesmente conexo e limitado do plano complexo.

Palavras-chave: superfícies de Riemann, Teorema da Uniformização, Teorema da Uniformização de Riemann, Circle packings, Teorema da Aplicação de Riemann, Análise Complexa, Variedades diferenciáveis

Abstract

The concept of differentiable manifolds and, in particular, surfaces is essential for the development of Geometry as we know it today.

As a matter of fact, we usually take into account the ambient space that is most familiar to us: $\mathbb{R}^2$ and develop a local theory around its established, proven and acquired theorems and general knowledge.

Nonetheless, investigating further, it is possible to introduce a complex structure on these surfaces, obtaining and a completely new theory adjacent to these objects: the theory of Riemann surfaces.

In this dissertation, our main purpose will be to dive ourselves in this concept, through its properties, theorems and generalizations of known results so that, in the end of the first part, we have a total categorization of all simply connected Riemann surfaces in a simple yet beautiful statement present in the Riemann Uniformization theorem.

Finally, in the last section, we shall also aboard new technique of approximating the bi-holomorphic mappings from simply connected, bonded and open sets of $\mathbb{C}$ given by the Riemann Uniformization theorem.

Keywords: Riemann surfaces, Uniformization theorem, Riemann Uniformization theorem, Riemann mapping theorem, Circle packings, Complex analysis, Differentiable manifolds

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Introduction

When considering differentiable manifolds, particularly, surfaces, one of the most interesting and marvelous results that come to mind is the famous theorem on the Classification of Compact Surfaces.

Delving deeper into this subject, a question that pondered through the minds of mathematicians was how to transform this result when considering a complex structure on manifolds (I will define what it is meant by this in the beginning of this dissertation). In other words, what are the possible classifications of Riemann surfaces? The answer comes on the form of the beautiful Riemann Uniformization Theorem:

Theorem 1 (Riemann Uniformization Theorem)**.** *Let M be a simply connected Riemann surface.*

Then, there exists a bijective holomorphic function $f : M \longrightarrow S$, where S is either the complex plane $\mathbb{C}$, the open unit disc $\mathbb{D}$ or the complex projective space $\mathbb{CP}(1)$.

Having in mind the work by Frakas and Kra on their book, [3], the first half of this dissertation shall have as its final objective the proof of the Uniformization Theorem, traversing through concepts such as one-forms, harmonic and holomorphic differentials, the assurance of their respective existence, the Dirichlet Problem and the characterization of every Riemann surface under just three categories: elliptic, hyperbolic and parabolic.

After having established this, an important and remarkable theorem of Complex Analysis emerges as a simple application: the Riemann Mapping Theorem 6.

From that point onward, we shall offer a new algorithm of approximating the Riemann maps given by this theorem using circle packings, a method inspired by the work of Rodin and Sullivan [2] but simultaneously offering new proofs for some theorems using results from the article [1], which help in simplifying many of the steps.

In the end, this shall present the reader with two new perspectives of the Riemann Uniformization theorem and its proof: one purely analytic and the second, a geometric view on one of its most fundamental corollaries.

1 Riemann Surfaces

We will begin by introducing the primary and most important notion of this text, namely the notion of *Riemann Surfaces*. Roughly speaking, a Riemann surface corresponds to (real) surfaces (or, equivalently, differentiable manifolds of dimension 2) equipped with an extra structure inducing on it a complex nature.

Definition 1. *Consider then a* 2*-dimensional real manifold* M. *A complex chart on* M *is a homeomorphism* $z : U \to V$, *where* U *is an open subset of* M *and* V *is a subset of* $\mathbb{C}$. *We will say that two complex charts* $z_i : U_i \to V_i$, $i = 1, 2$, *are holomorphically compatible if the map*

$$\theta_{21} = z_2 \circ z_1^{-1} : z_1(U_1 \cap U_2) \to z_2(U_1 \cap U_2)$$

is biholomorphic.

A homeomorphism $z : U \to V$ may also be called a *local coordinate* or a *local chart* on U. The term complex will be omitted most of times. Moreover, we say that a local coordinate $z : \mathcal{U} \longrightarrow \mathbb{C}$ is *centered at a point* $p \in M$ if $p \in \mathcal{U}$ and $z(p) = 0$. Notice that, for any $p \in M$ if $z : \mathcal{U} \longrightarrow \mathbb{C}$ is any local coordinate, then $\tilde{z} = z - z(p)$ will always be centered at the point p. With this notions in mind, we have the following.

Definition 2. *Let* M *be a real manifold of dimension* 2. *A complex atlas on* M *is a set of complex charts* $\mathcal{A} = \{(U_i, z_i)\}_{i \in I}$ *that are holomorphically compatible and such that* $\{U_i\}_{i \in I}$ *is an open cover of* M.

Imagine that a real manifold of dimension 2 is equipped with two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$. The two atlases are said analytically equivalent if the complex charts of $\mathcal{A}_1$ are holomorphically compatible with the complex charts of $\mathcal{A}_2$.

Next, fix an atlas $\{(\mathcal{U}_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ on M. Since the restriction of a complex chart z_α to an open subset of U_α is still a complex chart that is compatible with the initial one, we may say that we can always choose an atlas for M where the open sets $\mathcal{U}_\alpha$ are all simply connected and with bounded image. In this particular case, we say that each pair $(\mathcal{U}_\alpha, z_\alpha)$ is a *coordinate disk*.

Fix a complex atlas $\mathcal{A}$ on M. Denote by $\mathcal{A}^*$ the set constituted by all complex charts on M that are holomorphically compatible with $\mathcal{A}$. Then $\mathcal{A}^*$ is an atlas on M that, in addition, is compatible with any other chart compatible with $\mathcal{A}$. The atlas $\mathcal{A}^*$ is usually called a maximal atlas.

Definition 3. *A Riemann surface is a connected differentiable manifold of dimension* 2 *equipped with a maximal complex atlas* $\mathcal{A} = \{(\mathcal{U}_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$.

In the definition of Riemann surface we are asking for the existence of a maximal complex atlas. However, since to an atlas, there is always associated a maximal complex atlas, to prove that a certain 2-real dimensional manifold is a Riemann surface, it suffices to provide a complex atlas (maximal or not).

Before proceeding, let us thus present some examples of Riemann surfaces.

Example 1. The complex space $\mathbb{C}$ is the simplest and more natural example of a Riemann surface. In fact, $\{(\mathbb{C}, \mathrm{Id})\}$, where Id stands for the identity map on $\mathbb{C}$, is obviously an atlas for $\mathbb{C}$ satisfying the conditions in Definition 3.

Note that complex domains, *i.e.*, connected open subsets of $\mathbb{C}$ are also Riemann surfaces. Indeed, fix a domain $\mathcal{U} \subseteq \mathbb{C}$. An atlas giving to $\mathcal{U}$ the desired complex structure is $\{(\mathcal{U}, Id|_{\mathcal{U}})\}$. Particular examples, that will play a role in the sequel, include the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ as well as the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$, where $\mathfrak{Im}(z)$ stands for the imaginary part of z.

Example 2. The 2-Sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a Riemann surface. The well-know stereographic projections (both from north and south poles) correspond to the desire atlas. The proof that the maps of this atlas are holomorphic compatible can be found, for example, in [5].

Example 3. The projective space $\mathbb{CP}(1) = \mathbb{CP}^1$ also admits an atlas satisfying the conditions in Definition 3 and, thus, it is a Riemann surface as well. In fact, let $\{U_1, U_2\}$ be the open covering of $\mathbb{CP}(1)$, where $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}^* \cup \{\infty\}$. In addition, let z_1 stand for the identity map, while z_2 is given by $w \to \frac{1}{w}$. The transition function $\theta_{12} = z_1 \circ z_2^{-1}$ is clearly biholomorphic on $U_1 \cap U_2 = \mathbb{C}^*$.

It can easily be checked that the two Riemann surface S^2 and $\mathbb{CP}^1$ are isomorphic Riemann Surfaces (see [5]).

Example 4. Another example of a Riemann surface is the complex Torus. Consider then w_1, w_2, two complex numbers that are linearly independent over $\mathbb{R}$, and let T stand for the lattice spanned by w_1, w_2, *i.e.*,

$$T := \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2 = \{nw_1 + mw_2 : n,\, m \in \mathbb{Z}\}\,.$$

Next, on $\mathbb{C}$, we consider the equivalence classe where z is identified with z' if and only if $z - z' \in T$ and we denote by $\mathbb{C}/T$ its equivalence classes.

Denoting by $\pi : \mathbb{C} \to \mathbb{C}/T$ the canonical projection on $\mathbb{C}/T$, we may equip $\mathbb{C}/T$ with a topology with respect to which $\mathbb{C}/T$ is a Hausdorff topological space and the canonical projection π is continuous. More precisely, we will say that $U \subseteq \mathbb{C}/T$ is open on $\mathbb{C}/T$ if and only if $\pi(U)$ is an open subset of $\mathbb{C}$. The complex structure on $\mathbb{C}/T$ is then defined as follows

- fix a set V of $\mathbb{C}$ admitting no points equivalent under π and set $U = \pi(V)$. The map $\varphi = \pi^{-1} : U \to V$ is a complex chart on $\mathbb{C}/T$;

- consider the set of all charts as in the previous item. To prove that it constitutes an atlas, we need to check that all charts are holomorphically compatible.

  Fix then two charts (U_1, φ_1) and (U_2, φ_2). From the definition of φ_1 and φ_2 we have that for every $z \in \varphi_1(U_1 \cap U_2)$

$$\pi(\varphi_2 \circ \varphi_1^{-1}(z)) = \varphi_1^{-1}(z) = \pi(z)\,.$$

Having $\varphi_2 \circ \varphi_1^{-1}(z)$ and z itself the same projection, there follows that $\varphi_2 \circ \varphi_1^{-1}(z) - z \in T$, which means that the function

$$g(z) = \varphi_2 \,\circ\, \varphi_1^{-1}(z) - z$$

must be constant on every connected component of $\varphi_1(U_1 \cap U_2)$. Thus, the change of coordinates is, in particular, holomorphic.

Remark 1. If z_α and z_β are two local (complex) charts whose domains of definition has a non-empty overlapping, then $z_\alpha = \theta_{\alpha\beta} z_\beta$ for some holomorphic function $\theta_{\alpha\beta}$. The notation "$z_\alpha(z_\beta)$" may be used to express the relation in question.

1.1 Holomorphic and Conformal Functions

Let us begin this subsection by introducing the notion of holomorphic mappings between two Riemann surfaces.

Definition 4. *Let $f : M \longrightarrow N$ be a map between two Riemann Surfaces M and N. We say that f is a holomorphic mapping if there exist atlas $\{(\mathcal{U}_\alpha, z_\alpha)\}_{\alpha \in \Lambda_M}$ and $\{(\mathcal{V}_\beta, w_\beta)\}_{\beta \in \Lambda_N}$ for M and N, respectively, such that the function*

$$w_\beta \circ f \circ z_\alpha^{-1}$$

is holomorphic, as a function defined on an open subset of $\mathbb{C}$, for every $\alpha \in \Lambda_M$ and $\beta \in \Lambda_N$ such that the above map is defined, i.e., $\mathcal{U}_\alpha \cap f^{-1}(\mathcal{V}_\beta) \neq \emptyset$.

Note that, for the particular case where N coincides with $\mathbb{C}$, instead of holomorphic mapping we will talk about holomorphic function. The set of holomorphic functions defined on a Riemann Surface, over $\mathbb{C}$, will be denoted by $\mathcal{O}(M)$. Since a constant function is necessarily holomorphic and since both the sum and the product of holomorphic functions are holomorphic, we have that $\mathcal{O}(M)$ is a $\mathbb{C}$-algebra.

Definition 5. *A map $f : M \longrightarrow N$ between two Riemann surfaces M and N is called conformal or biholomorphic if it is bijective and holomorphic.*

Note that if a map is biholomorphic, its inverse map f^{-1} is automatically holomorphic.

Definition 6. *Two Riemann surfaces M and N are said to be conformally equivalent if there exists a conformal map $f : M \longrightarrow N$ between these two Riemann surfaces. If M and N are conformally equivalent, we will write $M \simeq N$.*

A well-known example of conformally equivalent Riemann Surfaces is the unit disk $\mathbb{D}$ and the complex upper half-plane $\mathcal{H}$. In fact, consider the map $\varphi : \mathcal{H} \to \mathbb{D}$ defined by

$$\varphi(z) = \frac{z-i}{z+i}\,.$$

The map φ is the restriction of a Möbius transformation to the upper-half plane, being clearly injective. We just need to show that it maps $\mathcal{H}$ onto $\mathbb{D}$. In other words, to do this, we just need to notice that

$$\begin{aligned}
|\varphi(z)| < 1 &\Leftrightarrow \left|\frac{z-i}{z+i}\right| < 1 \\
&\Leftrightarrow |z-i| < |z+i| \\
&\Leftrightarrow |z-i|^2 < |z+i|^2 \\
&\Leftrightarrow |z|^2 + iz - i\overline{z} + 1 < |z|^2 - iz + i\overline{z} + 1 \\
&\Leftrightarrow \mathfrak{Im}(z) > 0
\end{aligned}$$

and the conclusion follows.

An interesting observation to be done is the fact that it is possible for two surfaces to be diffeomorphic but not analytically equivalent. A simple example is the following. The unit disc $\mathbb{D}$ is diffeomorphic to the complex plane $\mathbb{C}$ via the map $z \mapsto \frac{z}{1-|z|}$. However, there is no bijective and conformal map between these two Riemann surfaces. In fact, suppose that $\varphi : \mathbb{C} \longrightarrow \mathbb{D}$ is a holomorphic map. Then, φ would be an entire and bounded function on $\mathbb{C}$. By Liouville Theorem, φ must be constant. Thus, surjectivity cannot be achieved.

As previously mentioned, there exist many results from complex analysis that also hold in the context of Riemann surfaces. Among these, we will state for the moment three of them, although we will provide the proof just for one. The other proofs are similar in the sense that all of them use local coordinates for the Riemann surfaces. We start by an analogue of the well-known identity theorem.

Theorem 2. *Let M and N be two connected Riemann surfaces. Let f_1 and f_2 be two holomorphic maps from M to N such that $\{p \in M : f_1(p) = f_2(p)\}$ has an accumulation points. Then f_1 and f_2 coincide everywhere.*

Proof. Denote by C the following set

$$C = \{p \in M : \exists U \,\text{neighborhood of}\; p \,:\, f_1|_U \equiv f_2|_U\}\,.$$

The set C is clearly an open subset of M. Let us next prove that C is closed as well.

Let then $q \in M$ be a point in the boundary of C. Since f_1 and f_2 are continuous, there immediately follows that $f_1(q) = f_2(q)$. Consider then local charts $z : Z \to V$ and $w : W \to V'$, where Z is an open subset of M containing q, W is an open subset o N

containing $f_1(q) = f_2(q)$ and both V and V' are open subsets of $\mathbb{C}$. We have that both

$$\overline{f}_1 = w \circ f_1 \circ z^{-1} : V \to V'$$
$$\overline{f}_2 = w \circ f_2 \circ z^{-1} : V \to V'$$

are holomorphic functions in the usual sense, as functions of $\mathbb{C}$. Furthermore, since the set of points where $\overline{f}_1$ and $\overline{f}_2$ coincide have q as an accumulation point, the Identity Theorem for holomorphic functions on domains in $\mathbb{C}$ implies that $\overline{f}_1$ and $\overline{f}_2$ coincide on V. There follows that f_1 and d_2 coincides on Z. We then conclude that C is closed.

Being C open and closed, there follows that C is either empty or coincides with M. It is easy to check that the accumulation point of the set defined in the statement of the Theorem belongs to C and, then, the result follows. ∎

By using atlases for Riemann surfaces, the following results can also be stated, as consequences of the Riemann removable singulatities Theorem and Liouville's Theorem, respectively.

Theorem 3. *Let M be a Riemann surface. Fix a point $p \in M$ and assume that $f \in \mathcal{O}(M \setminus \{p\})$ is bounded in a neighborhood of p. Then f admits a unique holomorphic extension to M.*

Theorem 4. *Let M be a compact and connected Riemann surface. Let $f \in \mathcal{O}(M)$. Then f must be constant.*

As an immediate consequence of Theorem 4 we may claim that neither $\mathbb{CP}^1$ nor the torus admit non-constant holomorphic function.

Let us finish this subsection by recalling the notion of meromorphic function and the notion of essential singularity of a function on Riemann surface.

Definition 7. *Let M be a Riemann surface and Υ a subset of M consisting of isolated points of M. We will say that f is a meromorphic function of M if f is a function defined* *from $M \setminus \Upsilon$ to $\mathbb{C}$ satisfying the following property*

$$\forall\, p_0 \in \Upsilon : \quad \lim_{p \to p_0} |f(p)| = \infty\,.$$

The points of Υ are called poles of f.

The set of meromorphic functions on a Riemann surface M will be denoted by $\mathcal{M}(M)$. Note that two functions $f,\, g \in \mathcal{M}(M)$ may have distinct set of poles. Furthermore, we may also note that a meromorphic function on a Riemann surface M can naturally be viewed as a holomorphic map between two Riemann surfaces, namely M and $\mathbb{CP}^1$.

Finally, we have the following.

Definition 8. *Let M be a Riemann surface and $p \in M$. In addition, let $f : M \setminus \{p\} \to \mathbb{C}$ be holomorphic. The point p is called an essential singularity for f is p is neither a pole of f nor a point over which f admits a holomorphic extension.*

1.2 Local Normal Form and Branching Number

For the moment, let us focus on holomorphic maps between Riemann surfaces. We start with a simple proposition.

Proposition 5. *Let M and N be two Riemann Surfaces and $F : M \longrightarrow N$ a non-constant holomorphic map. Fix a point $p \in M$. Then, there exist local coordinates $\phi : \mathcal{U} \to \mathbb{C}$ (resp. $\psi : \mathcal{V} \to \mathbb{C}$) centered at $p \in M$ (resp. $F(p) \in N$), and there exists a unique natural number $n \geq 1$ such that*

$$\psi \circ F \circ \phi^{-1}(z) = z^n$$

for every $z \in \phi(\mathcal{U}) \subseteq \mathbb{C}$.

Proof. Let then $\phi : \mathcal{U} \to \mathbb{C}$ be a local coordinate centered at p and $\varphi : \mathcal{V} \to \mathbb{C}$ a local coordinate centered at $F(p)$ and assume, without loss of generality, that $F(\mathcal{U}) \subseteq \mathcal{V}$. Since

F is holomorphic, there follows by definition that $\varphi \circ F \circ \phi^{-1}$ is a holomorphic function and thus, in particular, analytic at p. Since we are assuming the local coordinates to be centered at p and $F(p)$, we may write:

$$\varphi(F(\phi^{-1}(z))) = \sum_{k=n}^{\infty} a_k z^k \,,$$

for some natural number $n \geq 1$ and complex numbers $a_k \in \mathbb{C}$, for all $k \geq n$, with $a_n \neq 0$. The above expression can be re-written in the form

$$\varphi(F(\phi^{-1}(z))) = \sum_{k=n}^{\infty} a_k z^k = z^n \cdot \sum_{k=0}^{\infty} a_{n+k} z^k = z^n \cdot H(z) \,,$$

where $H(z)$ is a holomorphic function satisfying $H(0) = a_n \neq 0$. This function H is then an invertible function nearby the origin.

Naturally, it can be assumed that the open set $\mathcal{U}$, or equivalently, the open set $\phi(\mathcal{U})$ is sufficiently small (more precisely, a very small disc) so that $H = \exp^g$ for some holomorphic function g defined on the same open set. By letting $H = (e^{\frac{1}{n} \cdot g})^n$, there follows that there exists a holomorphic function $\tilde{g}$ such that $H = \tilde{g}^n$. Substituting this expression into the equation above, we get

$$\varphi(F(\phi^{-1}(z))) = z^n \cdot (\tilde{g}(z))^n = (z\tilde{g}(z))^n$$

By letting $R(z) = z\tilde{g}(z)$, we have $R'(0) = \tilde{g}(0) = a_n \neq 0$ and we use the Inverse Function Theorem to conclude that $R(z)$ is invertible in a neighborhood of 0, with a local inverse denoted by $\tilde{R}^{-1}$.

Now, $\psi := \tilde{R} \circ \phi$ is clearly a new local coordinate centered at p, and with respect to this new coordinate we have

$$
\begin{aligned}
\varphi(F(\psi^{-1}(z))) &= \varphi(F(\phi^{-1}(\tilde{R}^{-1}(z)))) \\
&= (\tilde{R}^{-1}(z) \cdot \tilde{g}(\tilde{R}^{-1}(z)))^n \\
&= (R \circ \tilde{R}^{-1})^n(z) \\
&= z^n
\end{aligned}
$$

Finally, concerning the uniqueness of the integer $n \geq 1$, it suffices to note that the value n represents the number of points in the pre-image of a generic point $p \in M$. This number is clearly a topological invariant an the result follows. ■

The number n defined in Theorem 5 for a holomorphic function F at a point p is called the *branching number of F at p* or *multiplicity of F at p*. This number will be and is denoted by $mult_p(F)$.

Thus, as a matter of curiosity, a holomorphic function having a branching number at p equal to 1 is equivalent to the local injectivity of f on a neighborhood of p. In particular, any local coordinate has a branching number equal to one.

2 Differentials on Riemann Surfaces

As in the case of differentiable manifolds M of real dimension m, we can associate to any Riemann surface a bundle of k-forms of its co-tangent budle: $\bigwedge^k TM^*$. Analogously to the real case, k-forms will be sections of the k-th exterior power of the cotangent bundle $T^*M = TM^*$

$$\bigwedge^k TM^* = \coprod_{x\in M} \bigwedge^k (T_xM)^*.$$

A greater emphasis to the case $k = 1$ of 1-forms on Riemann surfaces will be given along the text.

2.1 Differential Forms on Riemann Surfaces

Definition 9. *Let M be a Riemann surface.*

1. *A 0-form on M is a smooth function defined on M over $\mathbb{C}$.*
2. *A 1-form ω on M is a correspondence of an ordered pair of functions (f, g) to each local coordinate $z = x + iy$ and such that $\omega = f\,dx + g\,dy$ is invariant under change of coordinates.*

Let us precise what we mean to be invariant by change of coordinates. Consider on M two local coordinates $z = x + iy$ and $\tilde{z} = \tilde{x} + i\tilde{y}$ along with the corresponding pairs (f, g) and $(\tilde{f}, \tilde{g})$. Assume, in addition that the domains of the local coordinates in question have non-empty intersection. Then, we have:

$$\begin{pmatrix} \tilde{f} \\ \\ \tilde{g} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \tilde{x}} & \frac{\partial y}{\partial \tilde{x}} \\ \\ \frac{\partial x}{\partial \tilde{y}} & \frac{\partial y}{\partial \tilde{y}} \end{pmatrix} \begin{pmatrix} f(z(\tilde{z})) \\ \\ g(z(\tilde{z})) \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \mathfrak{Jac}(x,y)_{(\tilde{x},\tilde{y})} \end{pmatrix}^T \begin{pmatrix} f(z(\tilde{z})) \\ \\ g(z(\tilde{z})) \end{pmatrix}, \tag{2}$$

where $\mathfrak{Jac}(x,y)_{(\tilde{x},\tilde{y})}$ stands for the Jacobian matrix of the map $(\tilde{x},\tilde{y}) \mapsto (x(\tilde{x},\tilde{y}); y(\tilde{x},\tilde{y}))$.

Recall that the function $H = z \circ \tilde{z}^{-1} : \mathbb{C} \longmapsto \mathbb{C}$ is holomorphic and the partial derivatives of its real and imaginary parts with respect to $\tilde{x}\,, \tilde{y}$ are well defined.

More precisely, in this notation, "$\frac{\partial x}{\partial \tilde{x}}|_p$" represents the usual partial derivative relative to local coordinates: $\left.\frac{\partial(\mathfrak{Re}\, H)}{\partial x}\right|_{\tilde{z}(p)}$.

We should emphasize that if we define 1-forms originally as smooth sections of the co-tangent bundle $\bigwedge TM^*$, then, in each local coordinate $z : \mathcal{U} \to \mathbb{C}$, $z = x + iy$, we obtain a basis for this set: $\{dx, dy\}$. Thus, we write any 1-form, locally, as a linear combination: $f\,dx + g\,dy$, for some functions $f, g : \mathcal{U} \longrightarrow \mathbb{R}$. Later, in the chapter where I introduce the exterior derivative, the reader will have a better understanding about the matrix equation (1).

Definition 10. *Let M be a Riemann surface. A 2-form Ω on M is a correspondence, for each local chart $z = x + iy$, of a (real) function f such that:*

$$\Omega = f\,dx \wedge dy \tag{3}$$

invariant under coordinate changes.

Again, since we are assuming invariance by change of coordinates, a relation between f and $\tilde{f}$ on coordinates z and $\tilde{z}$, respectively, have a natural relation whenever the corresponding domains have non-empty intersection. More precisely, the following is satisfied:

$$\tilde{f} = f(z(\tilde{z})) \cdot \left|\frac{dz}{d\tilde{z}}\right|^2 = f \circ H \cdot \left|\frac{dH}{dz}\right|^2 \tag{4}$$

Once again, in (4), $\frac{dz}{d\tilde{z}}$ represents the (complex) derivative of $H = z \circ \tilde{z}^{-1}$, or equivalently, $\frac{dz}{d\tilde{z}} = \frac{1}{2}(\frac{dz}{d\tilde{x}} - i\frac{dz}{d\tilde{y}})$ and the remaining notation resembles that of 1-forms.
Now, we know that not all differentials have to be smooth or even differentiable.

Recall that given a differentiable manifold M we denote $\Omega^k M$ as the set of *differentiable k-forms* on M. In the case of a Riemann Surface M, the set of differentiable k-forms on M will also be denoted by $\Omega^k M$. We should, however, note that a 1-form (resp. 2-form) on a Riemann surface M is said to be differentiable if the functions f and g in the second point of Definition 9 (resp. f in Definition 10)) are differentiable for every local coordinate.

Next, I introduce the essential concept of Exterior Derivative $d : \Omega^k M \to \Omega^{k+1} M$, for all $k \in \mathbb{N}_0$, in the context of Riemann Surfaces:

Definition 11. *Let M be a Riemann surface. Consider first a smooth function f on M, i.e., a 0-form on M.*
The exterior derivative of a 0-form is the 1-form, defined in each local coordinate $z = x+iy$ as:

$$d(f) = df = f_x\, dx + f_y\, dy\,.$$

where f_x, f_y stands for the partial derivatives of f with respect to x, y, respectively.

Consider now a differentiable 1-form ω, and assume that it is written as $\omega = f dx + g dy$ in coordinates $z = x + iy$. The exterior derivative of ω is defined as the 2-form

$$d\omega := (g_x - f_y)\, dx \wedge dy$$

The expression of $d\omega$ can easily be deduced by asking the exterior derivative to be a linear operator. In fact, under this assumption, we have

$$\begin{aligned} d\omega &= d(f\,dx + g\,dy) \\ &= df \wedge dx + dg \wedge dy \\ &= (f_x\,dx + f_y\,dy) \wedge dx + (g_x\,dx + g_y\,dy) \wedge dy \\ &= (g_x - f_y)\,dx \wedge dy. \end{aligned}$$

Now let's clarify where the matrix expression in (1) arises. Initially, recall that if the domains of $\tilde{z}$ and z, say $\tilde{\mathcal{U}}$ and $\mathcal{U}$, respectively, intersect, then each is a holomorphic function of the other local coordinate. Thus, supposing in $\tilde{\mathcal{U}}$, a 1-form ω is given by $\omega = \tilde{f}\,d\tilde{x} + \tilde{g}\,d\tilde{y}$, then using the pullback through $\tilde{z}^{-1}$ at a point $p \in \mathbb{C}$:

$$\begin{aligned} (\tilde{z}^{-1})^*_p\,\omega &= \tilde{f} \circ \tilde{z}^{-1}(p)\,d(x \circ \tilde{z}^{-1}) + \tilde{g} \circ \tilde{z}^{-1}(p)\,d(y \circ \tilde{z}^{-1}) = \\ &= \tilde{f} \circ \tilde{z}^{-1}(p)\,d(\pi_1 \circ \tilde{z} \circ \tilde{z}^{-1}) + \tilde{g} \circ \tilde{z}^{-1}(p)\,d(\pi_2 \circ \tilde{z} \circ \tilde{z}^{-1}) = \\ &= \tilde{f} \circ \tilde{z}^{-1}(p)\,d\pi_1 + \tilde{g} \circ \tilde{z}^{-1}(p)\,d\pi_2 \end{aligned}$$

Where π_1 and π_2 represent the usual projections from $\mathbb{C}$ to $\mathbb{R}$ and it is omitted that the differentials are being taken on p.

Therefore, naming $H = z \circ \tilde{z}^{-1} = (H_1, H_2)$ and reminding ourselves of the rule on composition of pullbacks: $(f \circ g)^*_p\,\omega = g^*_p\,\left(f^*_{g(\cdot)}\omega\right)$, we conclude that:

$$
\begin{aligned}
(\tilde{z}^{-1})_p^* \omega &= H_p^* \left[(z^{-1})_{H(\cdot)}^* \omega \right] = \\
&= H_p^* \Big[f \circ z^{-1} \circ H \;\, d(x \circ z^{-1})_H + g \circ z^{-1} \circ H \;\, d(y \circ z^{-1})_H \Big] = \\
&= H_p^* \Big[f \circ \tilde{z}^{-1} (d\pi_1)_H + g \circ \tilde{z}^{-1} (d\pi_2)_H \Big] = \\
&= f \circ \tilde{z}^{-1} \circ H(p) \;\, (dH_1)_{H(p)} + g \circ \tilde{z}^{-1} \circ H(p) \;\, (dH_2)_{H(p)}
\end{aligned}
$$

Thus,

$$
\tilde{f} \circ \tilde{z}^{-1} = f \circ \tilde{z}^{-1} \circ H \cdot \frac{\partial H_1}{\partial x} + g \circ \tilde{z}^{-1} \circ H \cdot \frac{\partial H_2}{\partial x}
$$

and

$$
\tilde{g} \circ \tilde{z}^{-1} = f \circ \tilde{z}^{-1} \circ H \cdot \frac{\partial H_1}{\partial y} + g \circ \tilde{z}^{-1} \circ H \cdot \frac{\partial H_2}{\partial y}.
$$

Or, equivalently:

$$
f = \tilde{f} \circ (\tilde{z}^{-1} \circ z) \cdot \left. \frac{\partial H_1}{\partial x} \right|_{z(\cdot)} + \tilde{g} \circ (\tilde{z}^{-1} \circ z) \cdot \left. \frac{\partial H_2}{\partial x} \right|_{z(\cdot)}
$$

$$
g = \tilde{f} \circ (\tilde{z}^{-1} \circ z) \cdot \left. \frac{\partial H_1}{\partial y} \right|_{z(\cdot)} + \tilde{g} \circ (\tilde{z}^{-1} \circ z) \cdot \left. \frac{\partial H_2}{\partial y} \right|_{z(\cdot)}
$$

Therefore, if we associate the pair $(\tilde{f}, \tilde{g})$ to the local coordinate $(\tilde{\mathcal{U}}, \tilde{z})$, we necessarily obtain the equation (1). Similarly, we can conclude how the relation (4) was derived, with the additional help of the Cauchy-Riemann equations.

Now, making greater use of the complex structure and nature, we can also re-introduce the concept of the Exterior Derivative. I will always use the following notations (for a function f and $z = x + iy$ as a local coordinate):

$$\frac{\partial f}{\partial z} = f_z := \frac{1}{2}(f_x - i f_y)$$
$$\frac{\partial f}{\partial \overline{z}} = f_{\overline{z}} := \frac{1}{2}(f_x + i f_y)$$

$$dz = dx + i\,dy$$

$$d\overline{z} = dx - i\,dy$$

Following the complex notation, for a 0-form f in local coordinates z we have

$$df = f_z\,dz + f_{\overline{z}}\,d\overline{z}\,.$$

Additionally, for a 1-form ω locally written as $u\,dz + v\,d\overline{z}$ we have

$$d\omega = (v_z - u_{\overline{z}})\,dz \wedge d\overline{z}\,.$$

2.2 The Conjugation Operator

So far, we have only made use of concepts already known in (real) differentiable manifolds, but now we will introduce a new differential operator which is especially useful in the context of Riemann surfaces.

Let M be a Riemann surface. Initially, I make the following remark: the conjugation differential operator can be defined for k-forms, with $k = 0, 1, 2$, however, for this initial stage, we shall only discuss its application to 1-forms:

Definition 12. *Consider the* 1*-form* ω *given, in local coordinates* $z = x + iy$*, by* $\omega = f\,dx + g\,dy$*, where* f *and* g *are some functions defined on an open set of a Riemann* *surface. The conjugation operator $\star$ applied to ω is defined as the* 1*-form written locally as* ${}^\star\omega = -g\,dx + f\,dy$.

If the 1-form ω is given in complex notation by $\omega = u(z,\overline{z})\,dz + v(z,\overline{z})\,d\overline{z}$, then we will have that ${}^\star\omega = -i\,u(z,\overline{z})\,dz + i\,v(z,\overline{z})\,d\overline{z}$. Note, however, that, before proceeding, we need to check that the conjugation operator $\star$ is well defined in the sense that it does not depend on the local chart we choose. This is the content of the next proposition.

Proposition 6. *The conjugation operator $\star$ is well defined.*

Proof. Consider two local coordinates z and $\tilde{z}$ whose domains have non-empty intersection. Suppose the 1-form ω is given in coordinates $z = x + iy$ as $f\,dx + g\,dy$ and in coordinates $\tilde{z} = \tilde{x} + \tilde{y}$ as $\tilde{f}\,d\tilde{x} + \tilde{g}\,d\tilde{y}$. We know that the pairs (f,g) and $(\tilde{f},\tilde{g})$ satisfy the matrix relation (1), which means that

$$\tilde{f} = f x_{\tilde{x}} + g y_{\tilde{x}} \quad \text{and} \quad \tilde{g} = f x_{\tilde{y}} + g y_{\tilde{y}}$$

in the intersection of the domains.

By definition, the pairs of functions associated to the conjugation of ω in the coordinates z and $\tilde{z}$) are $(-g,f)$ and $(-\tilde{g},\tilde{f})$, respectively. In order to prove that the conjugation operation is well defined, we just need to check that these pair of functions satisfy the relation in Equation (1). We have

$$\begin{aligned} -g\,dx + f\,dy &= -g\,[x_{\tilde{x}}\,d\tilde{x} + x_{\tilde{y}}\,d\tilde{y}] + f\,[y_{\tilde{x}}\,d\tilde{x} + y_{\tilde{y}}\,d\tilde{y}] \\ &= [-g x_{\tilde{x}} + f y_{\tilde{x}}]\,d\tilde{x} + [-g x_{\tilde{y}} + f y_{\tilde{y}}]\,d\tilde{y}\,. \end{aligned}$$

Now, recalling that z is a holomorphic function of $\tilde{z}$, by using the Cauchy-Riemann equations we obtain that

$$\begin{aligned} -g\,dx + f\,dy &= (-gy_{\tilde{y}} - fx_{\tilde{y}})\,d\tilde{x} + (gy_{\tilde{x}} + fx_{\tilde{x}})\,d\tilde{y} \\ &= -\tilde{g}\,d\tilde{x} + \tilde{f}\,d\tilde{y} \end{aligned}$$

and the result follows. ■

Remark 2. There immediately follows from Definition 12 that the conjugation operation of the 1-forms dx and dy are given by

$$^\star dx = dy \quad \text{and} \quad ^\star dy = -dx\,.$$

The expression of the conjugation of the 1-forms for different "basis" of local coordinates is not necessarily so "simple". For example, consider the local coordinate $z = x + iy$ and the corresponding conjugate number $\overline{z} = \overline{x} - i\overline{y}$. The conjugation of dz is given by

$$^\star dz =^\star (dx + i\,dy) = -i\,dx + dy = -i\,d\overline{z}\,,$$

Consider now the the polar coordinates (r, θ), where $z = r\,e^{i\theta}$. In this case we have $^\star dr = r\,d\theta$ and $^\star d\theta = -r\,dr$. In fact, concerning the first equality and recalling that $r = \sqrt{x^2 + y^2}$ and $\theta = tg^{-1}(\frac{y}{x})$, we have

$$\begin{aligned}
{}^{\star}dr &= {}^{\star}(r_x\, dx + r_y\, dy) = -r_y\, dx + r_x\, dy \\
&= -r_y(x_\theta\, d\theta + x_r\, dr) + r_x(y_\theta\, d\theta + y_r\, dr) \\
&= -r_y(-r sin\theta\, d\theta + cos\theta\, dr) + r_x(r cos\theta\, d\theta + sin\theta\, dr) \\
&= -sin\theta\,(-r sin\theta\, d\theta + cos\theta\, dr) + cos\theta\,(r cos\theta\, d\theta + sin\theta\, dr) \\
&= r\, d\theta\,.
\end{aligned}$$

The second equality can be deduced in a similar way.

Recall that a differential 1-form ω is said to be *closed* if $d\omega = 0$. In turn, a 1-form ω is said to be *exact* if there exists a 0-form f such that $\omega = df$. The notions of exact and closed differential 1-forms can also be transferred to the 1-form obtained through the conjugation operator.

Definition 13. *A* 1*-form* ω *on a Riemann Surface is said to be co-closed if* ${}^{\star}\omega$ *is closed. Analogously, a* 1*-form* ω *on a Riemann surface is said to be co-exact if* ${}^{\star}\omega$ *is exact. (Equivalently, if* $\omega = {}^{\star}df$ *for some function* f*).*

Naturally, the notion of co-closed 1-form is only well defined for smooth differential 1-forms, *i.e.*, 1-forms $\omega = f\, dx + g\, dy$, with f and g smooth functions for every local coordinate $z = x + iy$. Many times we ask the 1-form (or the corresponding functions f and g) to be of class C^1 or even C^2.

2.3 Harmonic Functions and Differentials

Let f be a C^2 function on a Riemann surface.

Definition 14. *The Laplacian of a* C^2 *function* f *on a Riemann surface* M*, denoted by* Δf*, is the* 2*-form written, on each local coordinate* $z = x+iy$*, as* $\Delta f = (f_{xx} + f_{yy})\, dx \wedge dy$*. Furthermore, a function* f *is called harmonic if* Δf *vanishes identically, i.e.,* $\Delta f \equiv 0$.

Recall that if the Riemann surface is defined through an atlas with at least two local coordinates, we should then verify that the Laplacian is well-defined in the sense that it does not depend on the local coordinates we chose. This will immediately follow from the fact that $\Delta f = d^{\star} df$.

Example 5. An example of a harmonic function on $\mathbb{C}^{*}$ is $f(z) = \log|z| = \log(\sqrt{x^2+y^2}) = \frac{1}{2}\cdot\log(x^2+y^2)$. In addition, this example can be generalized in the following way. Suppose that f is a C^2 function on a Riemann surface M. If (U, φ) is local coordinate centered at point $p \in M$, then the function: $P \mapsto \log|\varphi(P)|$ from $U \backslash \{p\}$ to $\mathbb{C}$, usually represented by $log \mid \varphi \mid$, is also harmonic on a punctured neighborhood of p.

Next, by transferring the concept of harmonic 0-forms to differentials, we obtain the analogous concept of harmonic differentials. More precisely, we may define them in the following way.

Definition 15. *A* 1*-form* ω *is said to be harmonic if, locally, it is written as* $\omega = dh$ *for some harmonic function* h*.*

In other words, a harmonic differential ω is a collection of 1-forms $\{\omega_\phi\}$ for every local coordinate $(U, \phi = x+iy)$ in M such that $\omega_\phi = dh_\phi = \frac{\partial h_\phi}{\partial x}\,dx + \frac{\partial h_\phi}{\partial y}\,dy$, where h_ϕ are harmonic functions on U that are invariant through coordinate changes in the sense of (9).

Another family of 1-forms that we will focus on and study how it relates to the previously mentioned one, is the following.

Definition 16. *A* 1*-form* ω *is said to be holomorphic if, locally,* $\omega = df$ *for some holomorphic function* f*.*

More precisely, a holomorphic differential ω is a collection of 1-forms $\{\omega_\phi\}$ for every local coordinate (U, z_ϕ) in M such that $\omega_\phi = df_\phi = \frac{\partial f_\phi}{\partial z_\phi}\,dz_\phi$, where f_ϕ are holomorphic functions on U that are invariant through coordinate changes.

Remark. Consider two local coordinates z_α and z_β whose domains of definition have non-empty intersection. Consider next, a holomorphic 1-form ω on the union of the domain of the definition of the two coordinates. If $\omega = f_\alpha\, dz_\alpha$ and $\omega = f_\beta\, dz_\beta$ in coordinates z_α and z_β, respectively, then the invariance by change of coordinates lead us to the following relation: $f_\beta = f_\alpha \circ (z_\alpha^{-1} \circ z_\beta) \cdot \frac{\partial z_\alpha}{\partial z_\beta}$ on $U_\alpha \cap U_\beta$.

Let us verify some properties of each one of the special types of 1-forms defined above and see how they relate to each other.

Proposition 7. *A differential ω is harmonic if and only if it is closed and co-closed.*

Proof. Consider a differential 1-form ω and assume that ω is harmonic. Then, locally, $\omega = dh$ for some harmonic function h. This clearly implies that ω is close, *i.e.*, $d\omega = 0$ since $d^2 = 0$. Moreover, since it is harmonic, we have that $\Delta f = 0$ and, since $\Delta f = d^\star df = d^\star \omega$, we have that ω is co-closed as well.

For the converse, assume now that a 1-form ω is closed and co-closed. Being ω closed, we know that, locally, it takes on the form df for some smooth function f (cf. Poincaré's Theorem). In other words, $\omega = f_x\, dx + f_y\, dy$ in local coordinates. Using the fact that ω is co-closed, we have $d^\star\omega = d^\star df = \Delta f = 0$. Since this is true in any chart we chose, we conclude that ω is harmonic. ■

Proposition 8. *A differential ω is holomorphic if and only if it is closed and $^\star\omega = -i\omega$.*

Proof. Assume that the 1-form ω is holomorphic. We then have that, locally, $\omega = df$ for some holomorphic function f. Then $d\omega = 0$. There follows that, in each coordinate z, $\omega = df = f_z dz + f_{\overline{z}} d\overline{z}$. But, since f is holomorphic in the present coordinate, we have $f_{\overline{z}} = 0$. Therefore, $^\star\omega =^\star (f_z dz) = -i f_z dz = -i\omega$.

Assume now that we are given a closed 1-form satisfying $^\star\omega = -i\omega$. Since ω is closed, once again, in each local coordinate it is exact, i.e. $\omega = df$ for some function f at least C^2. Thus, $\omega = df = f_z dz + f_{\overline{z}} d\overline{z}$. However, since $^\star\omega = -i f_z dz + i f_{\overline{z}} d\overline{z}$ and

$-i\omega = -if_z dz - if_{\overline{z}} d\overline{z}$, we get that $if_{\overline{z}} = -if_{\overline{z}}$, i.e. $f_{\overline{z}} = 0$. This means that f is holomorphic and the proposition follows. ■

We finish this section with the following result.

Proposition 9. *A differential form ω is holomorphic if and only if $\omega = \alpha + i\,{}^\star\alpha$ for some harmonic differential α.*

Proof. Consider a differential 1-form ω and assume that it is holomorphic, Then ω can locally be written as $\omega = df = f_z dz$ foe some holomorphic function f. Let α be the 1-form defined as $\alpha = \frac{1}{2}(f_z dz + \overline{f}_{\overline{z}} d\overline{z})$. Then, through straightforward computations, we can verify that $\alpha + i\,{}^\star\alpha = \omega$. We just need to check that α is harmonic, i.e. that α is closed and co-closed. Note that

$$d\alpha = d\left[\frac{1}{2}(f_z dz + \overline{f}_{\overline{z}} d\overline{z})\right] = \frac{1}{2}\left(\overline{f}_{\overline{z}z} - f_{z\overline{z}}\right) dz \wedge d\overline{z} = 0$$

where the last equality follows from the fact that f is holomorphic. In fact, being f holomorphic, we have that $f_{z\overline{z}} = 0$, $\overline{f_z} = \overline{f}_{\overline{z}}$ and $\overline{f_{z\overline{z}}} = \overline{f}_{z\overline{z}} = \overline{f}_{\overline{z}z} = 0$. Similarly, we can check that ω is co-closed as well.

Assume now, that the α is a harmonic 1-form and let $\omega = \alpha + i\,{}^\star\alpha$. Assume, in addition that α is closed and co-closed. Then, $d\omega = 0$. By Proposition 8., it is enough to check that ${}^\star\omega = -i\omega$:

$${}^\star\omega = {}^\star(\alpha + i\,{}^\star\alpha) = {}^\star\alpha + i\,{}^\star({}^\star\alpha) = {}^\star\alpha - i\alpha = -i\omega\,.$$

and the result follows.

■

We have then characterized the harmonic and holomorphic differential 1-forms, as well as understood how they are related to each other. This will be useful in what follows.

3 The Space $\mathcal{L}^2$ of 1-forms

Let M be a Riemann surface and D a region of M. Consider a 1-form ω. We say ω a *measurable* 1-*form* if, when writing ω in each local coordinate z as $\omega = u\,dz + v\,d\overline{z}$, we have that u and v are measurable functions with respect to the coordinates. It is clear that from now on we identify two 1-forms if they coincide almost everywhere.

Imagine that we are given a measurable 1-form ω in a region D of a Riemann surface M. We define the norm of ω on D as

$$||\omega||_D = \iint_D \omega \wedge^\star \overline{\omega}\,.$$

Definition 17. *Let M be a Riemann surface and D a region of M. The space $\mathcal{L}^2(D)$ is the set formed by the* 1*-forms ω with finite norm on D, i.e., such that $||\omega||_D < \infty$.*

We have that $\mathcal{L}^2(D)$ equipped with the inner product $(.\,,.)_D$ defined as

$$(\omega_1, \omega_2)_D = \iint_D \omega_1 \wedge^\star \overline{\omega_2}\,,$$

where $\omega_1, \omega_2 \in \mathcal{L}^2(D)$, is a Hilbert space.

It is worth noting how the inner product behaves with respect to the conjugation operator: $({}^\star\omega_1, {}^\star\omega_2) = (\omega_1, \omega_2)$ and the local form of this inner product. Note, in addition, that on each local coordinate z, the expression on the integral is given by:

$$\begin{aligned}
\omega_1 \wedge^\star \overline{\omega_2} &= (u_1 dz + v_1 d\overline{z}) \wedge^\star \overline{(u_2\, dz + v_2\, d\overline{z})} \\
&= (u_1 dz + v_1 d\overline{z}) \wedge^\star (\overline{u_2}\, d\overline{z} + \overline{v_2}\, dz) \\
&= (u_1 dz + v_1 d\overline{z}) \wedge (-i\overline{v_2}\, dz + i\overline{u_2}\, d\overline{z}) \\
&= i(u_1\overline{u_2} + v_1\overline{v_2})\, dz \wedge d\overline{z}
\end{aligned}$$

After having introduced the Hilbert space $\mathcal{L}^2(D)$, we intend to study its structure and, later, to find a decomposition that encompasses the set of harmonic differentials. However, before that, we will provide a characterization of harmonic functions in $\mathcal{L}^2(D)$ as "weak" solutions of a certain integral equation. This corresponds to a result presented in Weyl's Theorem, which is discussed below.

3.1 Weyl's Lemma and the Decomposition of Harmonic Differentials

Let us then begin this subsection with the well-known Weyl's Lemma. To state it, let us just introduce the notion of *square-integrable function*. As before, let D be a region of a Riemann surface M. A measurable function f on D is *square-integrable* if

$$\int_D |f(z)|^2 \, dz < \infty \, .$$

In other words, $f \in \mathcal{L}^2(D)$.

Recalling that $\mathbb{D}$ stands for the open unit disk, *i.e.*, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we have the following.

Theorem 10 (Weyl's Lemma)**.** *Let φ be a square-integrable function in $\mathbb{D}$. We have that φ is harmonic (a.e. in $\mathbb{D}$) if and only if $\iint_{\mathbb{D}} \varphi \cdot \Delta\eta = 0$, for any smooth function η on $\mathbb{D}$ with compact support.*

Proof. Let φ be a square-integrable function in $\mathbb{D}$ and assume that φ is a harmonic. Fix a smooth function η on $\mathbb{D}$ with compact support. Thus, there exists $r \in (0, 1)$ such that *supp*(η) is contained in $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$.

To begin with, we have that $\int_{\partial\mathbb{D}_r} (\varphi^\star d\eta - \eta^\star d\varphi) = 0$, since η and, consequently, ${}^\star d\eta$

vanish on $\partial\mathbb{D}_r$. By using Stokes' Theorem (since η has compact support), we get that

$$\begin{aligned}
0 &= \int_{\partial\mathbb{D}_r} (\varphi^\star d\eta - \eta^\star d\varphi) \\
&= \iint_{\mathbb{D}_r} d(\varphi^\star d\eta - \eta^\star d\varphi) \\
&= \iint_{\mathbb{D}_r} (d\varphi \wedge^\star d\eta + \varphi d^\star d\eta - d\eta \wedge^\star d\varphi - \eta d^\star d\varphi) \\
&= \iint_{\mathbb{D}_r} (\varphi\Delta\eta - \eta\Delta\varphi)
\end{aligned}$$

Furthermore, being φ harmonic, we have that $\eta \cdot \Delta\varphi$ vanishes identically. Thus,

$$\iint_{\mathbb{D}} \varphi \cdot \Delta\eta = \iint_{\mathbb{D}_r} \varphi \cdot \Delta\eta = 0$$

For the converse, we assume initially that φ is a C^2 function in $\mathbb{D}$. Choose η to have compact support, *supp*(η), contained in $\mathbb{D}_r$. Then, we still obtain the previous equality:

$$\int_{\partial\mathbb{D}_r} (\varphi^\star d\eta - \eta^\star d\varphi) = \iint_{\mathbb{D}_r} (\varphi\, \Delta\eta - \eta\, \Delta\varphi)$$

Because of the properties we established for the support of η, we see that the first term above is zero, and thus the second is also.

But, by hypothesis, $\iint_{\mathbb{D}_r} \varphi\, \Delta\eta = 0$, so we have:

$$\iint_{\mathbb{D}_r} \eta\, \Delta\varphi = 0 = \iint_{\mathbb{D}} \eta\, \Delta\varphi\,,$$

for any smooth function η in $\mathbb{D}$ with compact support.

We claim that this suffices to conclude that $\Delta\varphi \equiv 0$ in $\mathbb{D}$:

To verify this, first, although both φ and η are complex functions, we can assume them to be real for the following argument. (Just separate into real and imaginary parts and apply it to each one.)

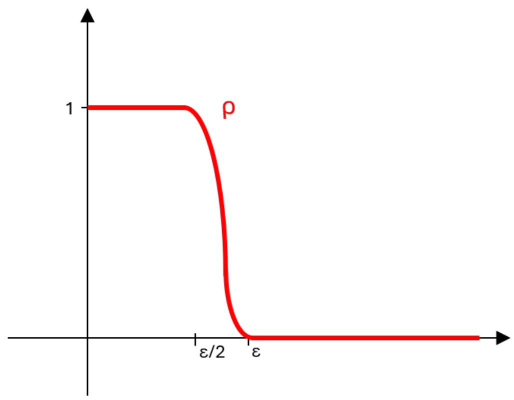


Figure 1: A graphic representation of ρ_ϵ.

Thus, assume that $\Delta\varphi \neq 0$ at a point $p \in \mathbb{D}$. Imagine that

$$\frac{\partial^2 \varphi}{\partial x^2}(p) + \frac{\partial^2 \varphi}{\partial y^2}(p) > 0\,,$$

by continuity, there will be a neighborhood U of p such that

$$\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)\bigg|_U > 0\,.$$

U can be chosen so that its closure is contained in $\mathbb{D}$.

Now, choose a function η of class C^2, positive and supported in U, and thus $\eta \cdot \Delta\varphi > 0$ and $\iint_{\mathbb{D}} \eta\ \Delta\varphi > 0$, a contradiction.

We conclude that $\Delta\varphi \equiv 0$ and φ is harmonic.

One could think the proof ends here, however, the statement does not assume ϕ is necessarily C^2, so we cannot make this assumption. We must proceed in order to circumvent this problem.

For this, consider the family of smooth functions $\rho_\epsilon : [0, \infty) \to \mathbb{R}$ such that $0 \leq \rho_\epsilon \leq 1$, $\rho_\epsilon = 0$ on $[\epsilon, \infty)$ and $\rho_\epsilon = 1$ on $[0, \epsilon/2]$.

And the slight modification: $\omega_\epsilon(r) = -\frac{1}{2\pi}\rho_\epsilon(r) \cdot log\ r$.

$$\gamma_\epsilon(z,w) = \begin{cases} 4 \cdot \frac{\partial}{\partial \bar{z} \partial z}\left(\omega_\epsilon(|z-w|)\right), & z \neq w \\ 0, & z = w \end{cases}.$$

Notice that, as $\rho_\epsilon = 0$ on $[\epsilon, \infty)$, $\gamma_\epsilon \equiv 0$ when $|z-w| > \epsilon$ and because a function f is harmonic if and only if $\frac{\partial f}{\partial \bar{z} \partial z} = 0$ (which implies, in this case, $log|z-w|$ is harmonic on $\mathbb{C}\backslash\{w\}$), besides the fact that $\rho_\epsilon = 1$ on $[0, \epsilon/2]$, we also have that $\gamma_\epsilon \equiv 0$ whenever $|z-w| < \epsilon/2$. Hence, $\gamma_\epsilon \equiv 0$ if $|z-w| < \epsilon/2$ or $|z-w| > \epsilon$.

Consider g a smooth function whose support is in $B(0, 1-2\epsilon)$ and the function

$$\psi(z) = \iint_{\mathbb{D}} \omega_\epsilon(|z-w|) g(w) \, \frac{dw \wedge d\overline{w}}{-2i} \,. \tag{5}$$

As a side note, notice that, if $w = x + iy$, this is:

$$\psi(z) = \iint_{\mathbb{D}} \omega_\epsilon(|z-w|) g(x,y) \; dx \wedge dy \,.$$

Let us briefly verify some properties of ψ.

Fist of all, ψ has compact support inside $B(0, 1-\epsilon)$. This is because if $w \in \mathbb{D}$ and $|z| \geq 1-\epsilon$, then $|w-z| \geq |w| - |z| \geq 1 - (1-\epsilon) = \epsilon$, so, as we saw above, $\omega_\epsilon(|w-z|) = 0$ and, as the integral is over $w \in \mathbb{D}$, it equals zero.

Secondly, notice that one could have integrated this expression over $\mathbb{C}$ instead of $\mathbb{D}$ and this would not affect the value of the integral since the support of g is inside $\mathbb{D}$.

Thirdly, applying the change of coordinates $w \mapsto z + w$, the integral on the function ψ above is the same as:

$$\psi(z) = \iint_{\mathbb{C}} \omega_\epsilon(|w|) g(w+z) \; dz \wedge d\overline{z} \,.$$

After this observations, let us proceed with derivation of ψ relative to $\overline{z}$:

$$\begin{aligned}\frac{\partial\psi}{\partial\overline{z}} &= \iint_{\mathbb{C}} \omega_\epsilon(|w|)\frac{\partial}{\partial\overline{z}}g(w+z)\; dz\wedge d\overline{z}\\ &= \iint_{\mathbb{C}} \omega_\epsilon(|w|)\frac{\partial}{\partial\overline{w}}g(w+z)\; dz\wedge d\overline{z}\\ &= \iint_{\mathbb{C}} \omega_\epsilon(|w-z|)\frac{\partial}{\partial\overline{w}}g(w)\; dz\wedge d\overline{z}\,.\end{aligned}$$

In an analogous way:

$$\frac{\partial\psi}{\partial z} = \iint_{\mathbb{C}} \omega_\epsilon(|w-z|)\frac{\partial}{\partial w}g(w)\; dz\wedge d\overline{z}.$$

Claim 1. We have that $4\frac{\partial^2\psi}{\partial z\partial\overline{z}} = -g + \iint_{\mathbb{C}} \gamma_\epsilon(z,w)g(w)\;\frac{dw\wedge d\overline{w}}{-2i}$.

We shall now split the integral in two pieces:

$$\begin{aligned}\psi(z) &= -\frac{1}{2\pi}\iint_{\{w:|z-w|\leq\epsilon/2\}} g(w)log|w-z|\;\frac{dw\wedge d\overline{w}}{-2i}\; + \iint_{\{w:|z-w|\geq\epsilon/2\}} g(w)\omega_\epsilon(|w-z|)\;\frac{dw\wedge d\overline{w}}{-2i}\\ &= \alpha(z)+\beta(z)\,.\end{aligned}$$

Proceeding with derivations as in the previous page, we know:

$$4\frac{\partial^2\beta}{\partial z\partial\overline{z}} = \iint_{\{w:|z-w|\geq\epsilon/2\}} \gamma_\epsilon(z,w)g(w)\;\frac{dw\wedge d\overline{w}}{-2i} = \iint_{\mathbb{C}} \gamma_\epsilon(z,w)g(w)\;\frac{dw\wedge d\overline{w}}{-2i}$$

In this way, we have proven one half of Claim 1., where the second equality derives from the fact that $\gamma_\epsilon(z,w)=0$ on $\{w:|z-w|\leq\epsilon/2\}$. Besides this, $\frac{\partial}{\partial z}(log|z-w|) = \frac{1}{2}\frac{1}{z-w}$ (the reader should be reminded one defined $\frac{\partial}{\partial z}$ as $\frac{1}{2}\cdot(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, so just write $|z-w| = \sqrt{(x-w_1)^2+(y-w_2)^2}$ and the calculations can be easily checked), so looking at the first piece of the integral:

$$\frac{\partial\alpha}{\partial z} = \frac{1}{4\pi}\iint_{\{w:|z-w|\leq\epsilon/2\}} \frac{g(w)}{w-z}\;\frac{dw\wedge d\overline{w}}{-2i}$$

We shall now analyze $4\frac{\partial^2\alpha}{\partial z\partial\bar{z}}$ and see it equals $-g$, through the following lemma:

Lemma 1 (Cauchy's Integral Formula)**.** *Suppose D is a domain in $\mathbb{C}$, bounded by a finite number of C^1 Jordan curves. If u is a smooth function on $\overline{D}$, then:*

$$2\pi i\, u(z) = \int_{\partial D} \frac{u(w)}{w-z} dw + \iint_D \frac{\frac{\partial u}{\partial \overline{w}}}{w-z}\, d\overline{w} \wedge dw\,.$$

Proof of Lemma 1. Choose a closed ball $B_\epsilon = \overline{B}(z,\epsilon) \subseteq D$ and let $D \backslash B_\epsilon$ be its complement on D. Then, using Stokes' Theorem:

$$\int_{\partial(D\backslash B_\epsilon)} \frac{u(w)}{w-z} dw = \iint_{D\backslash B_\epsilon} \frac{\partial}{\partial \overline{w}}\Big(\frac{u(w)}{w-z}\Big)\, dw \wedge d\overline{w} \qquad (\star)$$

The fist parcel is:

$$\int_{\partial(D\backslash B_\epsilon)} \frac{u(w)}{w-z} dw = \oint_{\partial D} \frac{u(w)}{w-z} dw - \oint_{B_\epsilon} \frac{u(w)}{w-z} dw$$

Using the parametrization of ∂B_ϵ: $\gamma(w) = z + \epsilon e^{it}$, $t \in [0, 2\pi]$ this becomes:

$$\int_{\partial(D\backslash B_\epsilon)} \frac{u(w)}{w-z} dw = \oint_{\partial D} \frac{u(w)}{w-z} dw - i\int_0^{2\pi} u(z+\epsilon e^{it}) dt$$

Finally, for the second parcel on $(\star)$ using the product rule for derivation it equals:

$$\iint_{D\backslash B_\epsilon} \frac{\partial u/\partial \overline{w}}{w-z}\, dw \wedge d\overline{w}.$$

Hence, the Cauchy's Integral Formula is demonstrated just by letting ϵ approach 0. ■

Continuing the proof, remember our objective is now to show the following:

Claim 2. We have that $4\,\frac{\partial^2\alpha}{\partial z\partial\bar{z}} = -g$.

Proof of Claim 2. In order to prove this Claim 2, one must first notice that we may assume (for fixed z) g has support on $|z-w|<\epsilon/2$ (which will help since, therefore, the integral expression for $\frac{\partial\alpha}{\partial z}$ can be taken all over $\mathbb{C}$ instead of just $|z-w|<\epsilon/2$). Let us verify this proposition.

Define $\nu_\epsilon(w,z)=\rho_\epsilon(2|z-w|)g(w)$. By the properties of ρ_ϵ, we know that $\nu_\epsilon = g$ on $\{w : |z-w|<\epsilon/4\}$ and that $\nu_\epsilon = 0$ on $\{w : |z-w|>\epsilon/2\}$. Also:

$$\begin{aligned}4\pi\frac{\partial\alpha}{\partial z} &= \iint_{\{w:|z-w|\leq\epsilon/2\}} \frac{g(w)}{w-z}\,\frac{dw\wedge d\overline{w}}{-2i} = \\ &= \iint_{\{w:|z-w|\leq\epsilon/2\}} \frac{\nu_\epsilon(w,z)}{w-z}\,\frac{dw\wedge d\overline{w}}{-2i} + \iint_{\{w:|z-w|\leq\epsilon/2\}} \frac{g(w)-\nu_\epsilon(w,z)}{w-z}\,\frac{dw\wedge d\overline{w}}{-2i}\,.\end{aligned}$$

The last integral represents a holomorphic function relative to the variable z.

Hence, the derivative relative to $\overline{z}$ of that parcel is zero and:

$$\begin{aligned}4\pi\cdot\frac{\partial^2\alpha}{\partial\overline{z}\partial z} &= \frac{\partial}{\partial\overline{z}}\left[\iint_{\{w:|z-w|\leq\epsilon/2\}} \frac{g(w)}{w-z}\,\frac{dw\wedge d\overline{w}}{-2i}\right] = \\ &= \frac{\partial}{\partial\overline{z}}\left[\iint_{\{w:|z-w|\leq\epsilon/2\}} \frac{\nu_\epsilon(w,z)}{w-z}\,\frac{dw\wedge d\overline{w}}{-2i}\right]\end{aligned}$$

So, for the proceeding calculations we shall use, without loss of generality, that g has support in $\{w\in\mathbb{C} : |w-z|<\epsilon/2\}$ and this assumption can, in fact, be done.

Going back to the computation of $\frac{\partial\alpha}{\partial z}$ and using once more the change of variable $w\mapsto w+z$ (and the last claim to extend the area of integration to $\mathbb{C}$), one concludes that

$$4\pi\frac{\partial\alpha}{\partial z} = \iint_{\mathbb{C}} \frac{g(w+z)}{w}\,\frac{dw\wedge d\overline{w}}{-2i}\,.$$

Thus

$$\begin{aligned} 4\pi\frac{\partial^2\alpha}{\partial\overline{z}\partial z} &= \iint_{\mathbb{C}} \frac{\partial g/\partial\overline{z}(w+z)}{w}\,\frac{dw\wedge d\overline{w}}{-2i} \\ &= \iint_{\mathbb{C}} \frac{\partial g/\partial\overline{w}(w+z)}{w}\,\frac{dw\wedge d\overline{w}}{-2i} \\ &= \iint_{\mathbb{C}} \frac{\partial g/\partial\overline{w}(w)}{w-z}\,\frac{dw\wedge d\overline{w}}{-2i} \end{aligned}$$

Remembering the support of g is in $|w-z|<\epsilon/2$, one could take the integral above to be the same as:

$$\iint_{|w-z|<\epsilon} \frac{\partial g/\partial\overline{w}(w)}{w-z}\,\frac{dw\wedge d\overline{w}}{-2i}$$

Applying the modification of Cauchy's Integral Formula seen before, the last integral is just saying: $-g(z) = 4\frac{\partial^2\alpha}{\partial\overline{z}\partial z}$. Thus Claim 2. and Claim 1. proofs are, simultaneously, finally finished. □

For the final steps on the proof of Weyl's Lemma, we use the condition on the statement with $\eta=\psi$ from (5). The fact that $\Delta\psi = 4\ \psi_{\overline{z}z}\,\frac{dz\wedge d\overline{z}}{-2i} = 4\ \psi_{z\overline{z}}\,\frac{dz\wedge d\overline{z}}{-2i}$ and all the conclusions we have made until, *i.e.*, $\psi=\alpha+\beta$, $4\frac{\partial^2\alpha}{\partial\overline{z}\partial z} = -g$ and $4\frac{\partial^2\beta}{\partial z\partial\overline{z}} = \iint_{\mathbb{C}} \gamma_\epsilon(z,w)g(w)\,\frac{dw\wedge d\overline{w}}{-2i}$ in the following way:

$$0 = \iint_{\mathbb{D}} \varphi(z)\cdot\Delta\psi = -\iint_{\mathbb{D}} \varphi\cdot\Big(g\,\frac{dz\wedge d\overline{z}}{-2i}\Big) \;+\; \iint_{\mathbb{D}} \varphi\cdot\Big(4\frac{\partial^2\beta}{\partial z\partial\overline{z}}\,\frac{dz\wedge d\overline{z}}{-2i}\Big)$$

So, using Fubini's Theorem in order to change the order of the variables we get

$$\begin{aligned} \iint_{\mathbb{D}} \varphi(z)g(z)\,\frac{dz\wedge d\overline{z}}{-2i} &= \iint_{\mathbb{D}} \varphi(z)\cdot\Big(\iint_{\mathbb{C}} \gamma_\epsilon(z,w)g(w)\,\frac{dw\wedge d\overline{w}}{-2i}\Big)\,\frac{dz\wedge d\overline{z}}{-2i} = \\ &= \iint_{\mathbb{D}} \varphi(z)\cdot\iint_{\mathbb{D}} \gamma_\epsilon(z,w)g(w)\,\frac{dw\wedge d\overline{w}}{-2i}\,\frac{dz\wedge d\overline{z}}{-2i} = \\ &= \iint_{\mathbb{D}} g(w)\cdot\iint_{\mathbb{D}} \gamma_\epsilon(z,w)\varphi(z)\,\frac{dz\wedge d\overline{z}}{-2i}\,\frac{dw\wedge d\overline{w}}{-2i} \end{aligned}$$

By letting $G_\varphi(w) = \iint_{\mathbb{D}} \gamma_\epsilon(z,w)\varphi(z)\, \frac{dz\wedge d\overline{z}}{-2i}$, we can rewrite the above as:

$$\iint_{\mathbb{D}} g(w)\cdot\varphi(w)\,\frac{dw\wedge d\overline{w}}{-2i} = \iint_{\mathbb{D}} g(w)\cdot G_\varphi(w)\,\frac{dw\wedge d\overline{w}}{-2i}\,,$$

for any smooth function g with support on $B(0, 1-2\epsilon)$ and any $\epsilon > 0$ (or, simpler words, for any smooth function g with support on $\mathbb{D}$).

Now, using the fact that smooth functions with compact support are dense in $\mathcal{L}^2(\mathbb{D})$, we can be able to prove that actually $G_\varphi \equiv \varphi$, *a.e.*, in $\mathbb{D}$.

Let us demonstrate this, denoting $L = G_\varphi - \varphi$.

Start choosing a function g such that $\iint_{\mathbb{D}} |g-L|^2\, \frac{dw\wedge d\overline{w}}{-2i} \leq \epsilon$, then:

$$\begin{aligned}
\iint_{\mathbb{D}} |g-L|^2\,\frac{dw\wedge d\overline{w}}{-2i} &= \iint_{\mathbb{D}} |g|^2\,\frac{dw\wedge d\overline{w}}{-2i} - \iint_{\mathbb{D}} (\overline{g}L + g\overline{L})\,\frac{dw\wedge d\overline{w}}{-2i} + \iint_{\mathbb{D}} |L|^2\,\frac{dw\wedge d\overline{w}}{-2i}\\
&= \iint_{\mathbb{D}} |g|^2\,\frac{dw\wedge d\overline{w}}{-2i} + \iint_{\mathbb{D}} |L|^2\,\frac{dw\wedge d\overline{w}}{-2i}\\
&\leq \epsilon
\end{aligned}$$

In particular, $\iint_{\mathbb{D}} |L|^2\, \frac{dw\wedge d\overline{w}}{-2i} \leq \epsilon$, $\forall \epsilon > 0$, hence, $\iint_{\mathbb{D}} |L|^2\, \frac{dw\wedge d\overline{w}}{-2i} = 0$ and $L \equiv 0$ *a.e.* in $\mathcal{D}$.

This way, one concludes:

$$\varphi(w) = \iint_{\mathbb{D}} \gamma_\epsilon(z,w)\,\varphi(z)\,\frac{dz\wedge d\overline{z}}{-2i},\ a.e. \text{ in } \mathcal{D}.$$

And, since the right-hand side is always C^∞ (with respect to w), φ is smooth and Weyl's Lemma proof is finished. ∎

After establishing this result, we may now begin the first steps to find the aforementioned decomposition of the space $\mathcal{L}^2(M)$ into closed subspaces, starting by defining the following sets:

Definition 18. *Let E and $E^\star$ the spaces:*

$$E = \overline{\{\, df : f \text{ is a smooth function on } M \text{ with compact support} \}} \subseteq \mathcal{L}^2(M)$$

and

$$E^\star = \{\, \omega \in \mathcal{L}^2(M) : {}^\star\omega \in E \,\} \subseteq \mathcal{L}^2(M)$$

Explaining further, the space E is the closure, with respect to the topology (and norm) of $\mathcal{L}^2(M)$, of the set of exact 1-forms coming from smooth functions in M compactly supported and the second case is an analogous set using the conjugation operator.

By definition, $\omega \in E$ (resp. $E^\star$) if and only if there exists a sequence $(f_n)_n$ of smooth functions in M with compact support such that $\omega = \lim_n df_n$ (resp. $\omega = \lim_n {}^\star df_n$).

Since $\mathcal{L}^2(M)$ is a Hilbert space with inner product equal to $(\cdot\,,\cdot)$, it is clear that we may consider their *orthogonal complements* which are respectively given by

$$E^\perp = \{\omega \in \mathcal{L}^2(M) : \ (\omega,\, \alpha) = 0,\ \forall\, \alpha \in E\}$$

and

$$\begin{aligned} E^{\star\perp} &= \{\omega \in \mathcal{L}^2(M) : \ (\omega,\, \alpha) = 0,\ \forall\, \alpha \in E^\star\} \\ &= \{\omega \in \mathcal{L}^2(M) : (\omega,\, {}^\star df) = 0, \forall\, C^\infty \text{ functions } f \text{ in } M \text{ with compact support}\} \end{aligned}$$

Proposition 11. *Let $\alpha \in \mathcal{L}^2(M)$ be a class C^1 differential. Then $\alpha \in E^{\star\perp}$ (resp. $E^\perp$) if and only if α is closed (resp. co-closed).*

Proof. Let α be closed. Let f be smooth and with compact support inside $\mathcal{K}$, a set with compact closure on M. In these conditions, we can apply Stokes' Theorem and obtain:

$$(\alpha, {}^\star df) = -\iint_M \alpha \wedge \overline{df} = -\iint_{\mathcal{K}} \alpha \wedge \overline{df} = -\iint_{\mathcal{K}} d(\overline{f}\alpha) - d\alpha \wedge \overline{f} = -\iint_{\mathcal{K}} d(\overline{f}\alpha)\,.$$

The last equality follows from $d\alpha = 0$. Finally, by Stokes' Theorem:

$$-\iint_{\mathcal{K}} d(\overline{f}\alpha) = -\int_{\partial\mathcal{K}} \overline{f}\alpha = 0$$

since, by the support of f, the function vanishes on $\partial\mathcal{K}$. Thus, $\alpha \in E^{\star\perp}$.

Now, assume $\alpha \in E^{\star\perp}$. For all smooth functions f with compact support on M, by definition $\iint_M \alpha \wedge \overline{df} = 0$, and similarly to what we did for the previous implication, $\iint_M d(\overline{f}\alpha) = 0$, thus, we conclude that: $-\iint_M d(\overline{f}\alpha) = \iint_M \alpha \wedge \overline{df} - d\alpha \wedge \overline{f} = -\iint_M \overline{f}\, d\alpha = 0$, for all smooth functions f on M with compact support.

As a consequence, $d\alpha = 0$, *i.e.*, it is a closed form. ■

From this characterization, we conclude that E and $E^\star$ are orthogonal to each other (*i.e.*, $E \subseteq E^{\star\perp}$ since exact forms are closed and vive-versa). Now, since the space $\mathcal{L}^2(M)$ is a Hilbert space and the subspaces E and $E^\star$ are closed and orthogonal to each other, $E \cap E^\star \subseteq (E^\star)^\perp \cap E^\star = (0)$, $E \oplus E^\star$ is well defined and we may consider the orthogonal decomposition: $\mathcal{L}^2(M) = (E \oplus E^\star) \oplus (E \oplus E^\star)^\perp$.

Let us further analyze the orthogonal complement: $(E \oplus E^\star)^\perp$, which I will denote from now on as $\mathcal{H}$, so that $\mathcal{L}^2(M) = (E \oplus E^\star) \oplus \mathcal{H}$.

An universal property of Hilbert spaces guarantees that $\mathcal{H} = (E \oplus E^\star)^\perp = E^\perp \cap E^{\star\perp}$, which might immediately lead us to think that $\mathcal{H}$ is the intersection of the closed and co-closed 1-forms, *i.e.*, the harmonic forms (thanks to Propositions 11 and 7). We will see that this is, in fact, true. However, it is not as straightforward as one might think initially. Recall that Proposition 11 requires differentiability conditions (class C^1) that we do not necessarily have in the space $\mathcal{H} \subseteq \mathcal{L}^2(M)$, where we only have assurance of measurability. This transition should remind us of the previously mentioned Weyl's Lemma, and indeed, this deduction shall be its greatest utility. But before doing that, let us look at one more property which relates all the subsets studied until now:

Proposition 12. *$E^{\perp} = E^{\star} \oplus \mathcal{H}$ and $E^{\star\perp} = E \oplus \mathcal{H}$.*

Proof. We know that: $\mathcal{L}^2(M) = (E \oplus E^{\star}) \oplus (E \oplus E^{\star})^{\perp} = E \oplus E^{\star} \oplus \mathcal{H} = E \oplus \{E^{\star} \oplus \mathcal{H}\}$, but it is also clear that $\mathcal{L}^2(M) = E \oplus E^{\perp}$. Thus, we have $E^{\perp} = E^{\star} \oplus \mathcal{H}$. Similarly for $E^{\star\perp}$.

■

We are now ready to prove that $\mathcal{H}$ is indeed the space of harmonic functions in $\mathcal{L}^2(M)$:

Theorem. *$\mathcal{H}$ is equal to the set of harmonic 1-forms in $\mathcal{L}^2(M)$.*

Proof. If ω is harmonic, then it is both closed and co-closed, so by Proposition 11, it belongs to $E^{\star\perp}$ and $E^{\perp}$. Thus, $\omega \in \mathcal{H}$.

Now, assume $\omega \in \mathcal{H}$. (Once again, I remind that we cannot immediately infer that ω is of class C^2). Let D be a coordinate disk in M with coordinates z and consider η a smooth real function with (compact) support in D.

We write in this disk $\omega \in \mathcal{L}^2(M)$ as $p\,dx + q\,dy$, where p and q are (only) measurable. Since $\omega \in E^{\perp} \cap E^{\star\perp}$, we have $(\omega, d\eta_x) = (\omega, {}^{\star} d\eta_y) = 0$ and:

$$
\begin{aligned}
0 = (\omega, d\eta_x -^{\star} d\eta_y) &= (p\,dx + q\,dy, d\eta_x -^{\star} d\eta_y) = \iint_D (p\,dx + q\,dy) \wedge^{\star} (\overline{d\eta_x -^{\star} d\eta_y}) \\
&= \iint_D (p\,dx + q\,dy) \wedge (\eta_{xx} + \eta_{yy})\,dy = \iint_D p \cdot (\eta_{xx} + \eta_{yy})\,dx \wedge dy = \iint_D p\,\Delta\eta
\end{aligned}
$$

Similarly, since ${}^{\star}\omega$ also belongs to $E^{\perp} \cap E^{\star\perp}$, we conclude that $\iint_D q\,\Delta\eta = 0$.

By Weyl's Lemma, we can conclude that both p and q are harmonic and at least of class C^1, so finally, we can safely invoke Proposition 11 and assert that under these conditions, since $\omega \in E^{\perp} \cap E^{\star\perp}$, it is closed and co-closed, *i.e.*, harmonic.

■

Thus, we finish our digression through the $\mathcal{L}^2(M)$ space of 1-forms. I will leave just one last interesting remark that can be easily concluded from these results: a compact Riemann surface does not have non-constant harmonic functions. Notice that, if we consider M as a compact Riemann surface, then the set E coincides with the exact 1-forms (this does not generally occur). Moreover, if h is a harmonic function, then $dh \in E \cap \mathcal{H}$, so $dh \equiv 0$ and h is constant (remember every Riemann surface is connected by assumption) .

Note, however, that there may exist harmonic differentials which are not identically zero on compact Riemann surfaces.

Naturally, new questions arise, one of them being: is it possible to always construct harmonic functions on a Riemann surface if one allows for singularities to exist at some points? The answer comes in the form of the following section.

3.2 Existence of Harmonic Functions

We now know that the answer to the existence of harmonic functions in the compact case is limited to constants. Let's recall the case of holomorphic complex functions. A "generalization" of them would be the meromorphic functions, *i.e.*, holomorphic but with a finite number of singularities (poles).

Thinking in a similar way, we can consider harmonic functions with singularities. Starting from these, the answer to the question of their existence will be affirmative, as we'll see in this chapter.

Theorem. *Let $n \in \mathbb{N}$, M be a Riemann surface, p a point on M and z a local coordinate vanishing at p. Then, there exists a function u on M that satisfies:*

(i) u is harmonic on $M \backslash \{p\}$;

(ii) $u - z^{-n}$ is harmonic in a neighborhood N of p;

(iii) u satisfies

$$\| \, du \, \|_{\mathcal{L}^2(M \setminus N)} = \iint_{M\setminus N} du \wedge^{\star} \overline{du} \quad < \infty \ ;$$

(iv) $(du, df) = 0 = (du,^{\star} df)$ for every function f on M with compact support, vanishing in a neighborhood of p.

Proof. Let D be a coordinate disk with local coordinates $z = x + iy$ such that $z(p) = 0$. Assume z is a local coordinate with image $2 \cdot \mathbb{D}$, the open disk $\{z \in \mathbb{C} : |z| < 2\}$.

Initially, we consider a function $h : M \to \mathbb{C}$ with a singularity at p under the conditions (which will be called Regularity Conditions) that: h is harmonic in $\{P \in D : \ \frac{1}{2} \leq |z(P)| \leq 1\}$ and $^{\star}dh = 0$ in $\{P \in D : \ |z(P)| = 1\}$. We abbreviate this set as $\{|z| = 1\}$.

An example of a function under these conditions is given, in $D \cap \{P \in M : |z(P)| \leq 1\}$, by:

$$h(P) = z^{-n}(P) + \overline{z}^{n}(P).$$

With h extended to be zero outside that region.

(To simplify the writing, we will identify each point in M by its image using the local coordinate z, and thus write h as $h(z) = z^{-n} + \overline{z}^{n}$.)

First of all, it is simple to verify that h is harmonic on $\{z : 1/2 < |z| < 1\}$, since

$$-2i \, \frac{\partial^2 h}{\partial z \partial \overline{z}} \, dz \wedge d\overline{z} = \Delta h,$$

and it is clear that differentiating h first with respect to $\overline{z}$ and then with respect to z results in 0.

To see that $^{\star}dh = 0$ on $\{z : \ |z| = 1\} = \partial D$, we consider the restriction $\tilde{h}$ of h to this set: $\tilde{h} = h|_{\partial D}$.

It is clear that $i^{*}(dh) = d\tilde{h} = d(h|_{\partial D})$ (where i is the inclusion $i : \partial D \longrightarrow D$ and i^{*} is the pullback restricting the tangent space), and to obtain the above equality, we use polar coordinates (r, θ) instead of the usual (x, y) on the local coordinate (D, z), so that:

$$h(r,\theta) = \frac{1}{r^n} e^{-in\theta} + r^n e^{-in\theta},$$

and thus,

$$\tilde{h}(r,\theta) = 2e^{-in\theta} \quad \text{(which does not depend on } r\text{, as expected).}$$

Therefore,

$$d\tilde{h} = \tilde{h}_\theta \, d\theta + \tilde{h}_r \, dr = \tilde{h}_\theta \, d\theta$$

and

$$^\star d\tilde{h} = -r\tilde{h}_\theta \, dr.$$

Since the function $r : P \in \partial D \mapsto |z(P)|$ is constant and equal to 1 on ∂D, we have $dr = 0$.

Thus, $^\star d\tilde{h} = 0$, *i.e.*, $^\star dh = 0$ on $\{P \in D : \ |z(P)| = 1\}$.

Now, we construct a function $H : M \to \mathbb{C}$ given by $H \equiv h$ in $\frac{1}{2} \leq |z| \leq 1$, equal to 0 outside $|z| \leq 1$, and imposing that H is smooth in $\{\, |z| \leq 1 \,\}$ (making use of the partitions of unity). In summary, visually:

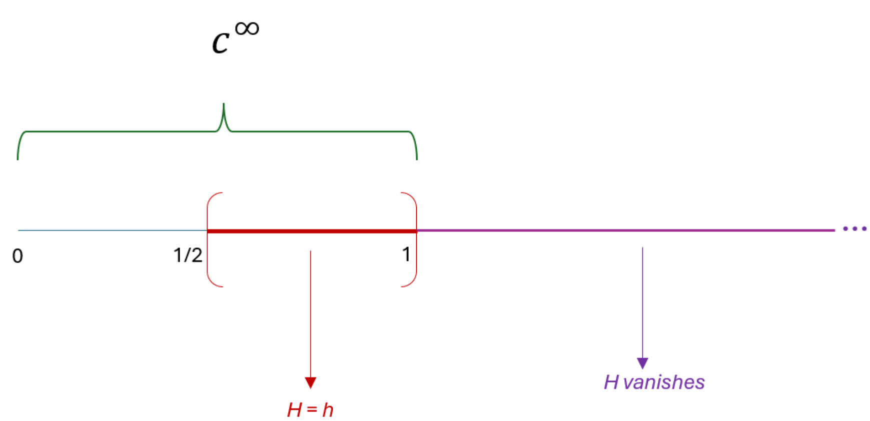


Figure 2: A diagram resuming the construction of H.

Thus, H is C^∞ on M, except on a null-measure set: $\partial D = \{\, |z| = 1 \,\}$.

Next, we form the differential dH, which is well-defined except for a null-measure set, something that will not cause any problems because we are interested (only) in measurable differentials. Accordingly, henceforth, we shall assume that dH is well-defined throughout M and $dH \in \mathcal{L}^2(M)$, even though, it is not a smooth differential over M. Thus, we are not allowed to use the good properties shared by these and, initially, must restrict ourselves to results from $\mathcal{L}^2(M)$ strictly.

Continuing, we write $dH = \alpha + \omega$, with $\alpha \in E$ and $\omega \in E^\perp = E^\star \oplus \mathcal{H}$.

Let us now see that α is harmonic in $M \setminus \overline{D_{1/2}}$, *i.e.*, $\alpha \in E^\perp \cap (E^\star)^\perp \subseteq \mathcal{L}^2(M \setminus \overline{D_{1/2}})$.

Now, we shall first explore the behaviour of α in a general context, that is, its properties in any local coordinate of M, in order to later, manipulate this local coordinate as we may please:

Let $(\mathcal{U}, \epsilon = \xi + i\eta\,)$ be a conformal disk covering a point Q such that $\epsilon(Q) = 0$.

Consider a real function ρ smooth on M with compact support in $\mathcal{U}$. We write the 1-form α as $p\, d\xi + q\, d\eta$ in $\mathcal{U}$.

Consider also ρ_ξ and ρ_η, since $\alpha \in E^{\star\perp}$ (Proposition 12), then: $0 = (\alpha, {}^\star d\rho_\eta) = \iint_{\mathcal{U}} (-p\, \rho_{\eta\eta} + q\, \rho_{\eta\xi})\ d\xi \wedge d\eta$.

Now, $(\alpha, d\rho_\xi) {=} \iint (p\, \rho_{\xi\xi} + q\, \rho_{\xi\eta})\ d\xi \wedge d\eta$ and because $\omega \in E^\perp$, we can see this equals $(dH, d\rho_\xi)$. Thus:

$$(dH, d\rho_\xi) = (dH, d\rho_\xi) - 0 = \iint p(\rho_{\eta\eta} + \rho_{\xi\xi})\, d\xi \wedge d\eta = \iint p \cdot \Delta\rho$$

Similarly, making use of ρ_ξ instead of ρ_η and summing, we obtain the equality: $(dH, d\rho_\eta) = \iint q \cdot \Delta\rho$.

Claim 1. *α is harmonic on $M \backslash \overline{D_{1/2}}$.*

Select a triple $(\mathcal{U}, \epsilon, \rho)$ on the previously described conditions, such that $\mathcal{U} \subseteq M \backslash \overline{D_{1/2}}$. Thanks to arguments mentioned above, $\iint_{\mathcal{U}} p \cdot \Delta\rho = (dH, d\rho_\xi)_M$.

Our main purpose has now changed to be applying Weyl's Lemma. Hence, because of what we saw in the beginning, we must guarantee that $\iint_{\mathcal{U}} p \cdot \Delta\rho = (dH, d\rho_\xi)_M = 0$. We know H vanishes on the complement of D and ρ has support inside $\mathcal{U} \subseteq M\backslash\overline{D_{1/2}}$, thus, $(dH, d\rho_\xi)_D = 0$ and $(dH, d\rho_\xi)_{M\backslash D_{1/2}} = 0$.

All that remains is to verify $(dH, d\rho_\xi)_{D\backslash D_{1/2}} = 0$ in order to conclude $(dH, d\rho_\xi)_M = 0$.

Claim 2. $(dH, d\rho_\xi)_{D\backslash D_{1/2}} = 0$:

We may, with no loss of generality (and to make calculations easier), also assume that $\mathcal{U} \subseteq D$ and, hence, that the first local coordinates we used, z, are equal to ϵ.

Proceeding with calculus of the integral over the local coordinate z, Stokes Theorem tells us that:

$$(d\rho_x, dH)_{D\backslash D_{1/2}} = \iint_{D\backslash D_{1/2}} d\rho_x \wedge {}^\star(d\overline{H}) = -\iint_{D\backslash D_{1/2}} \rho_x \cdot \Delta\overline{H} + \int_{\partial D} \rho_x \, {}^\star d\overline{H} - \int_{\partial D_{1/2}} \rho_x \, {}^\star d\overline{H}$$

Let us analyze each parcel: it is clear that the first one equals zero as H is harmonic on $D\backslash D_{1/2}$ since, there, $H \equiv h$. The middle expression is also zero since ${}^\star d\overline{H} = {}^\star d\overline{h} = 0$ when we restrict H on ∂D (as we have seen before) and $supp(\rho) \subseteq M\backslash\overline{D_{1/2}}$, therefore, $\rho|_{\partial D_{1/2}} = 0$.

Therefore, $(dH, d\rho_x)_M = (\alpha, d\rho_x)_M = \iint_{\mathcal{U}} p \cdot \Delta\rho = 0$, for all smooth functions ρ with compact support in $M\backslash\overline{D_{1/2}}$ and this ends the (sub-)Claim 2.. $\square$

Thus, by Weyl's Lemma, p is harmonic in $M\backslash\overline{D_{1/2}}$. Analogously, using a similar argument, just changing the role of $d\rho_x$ by $d\rho_y$, we come to the conclusion that $\iint_{\mathcal{U}} q \cdot \Delta\rho = 0$. So, q is also harmonic in $M\backslash\overline{D_{1/2}}$.

In particular, α is class C^1. Thus, as $\alpha \in E^{\star\perp}$, by Proposition 11, α is a closed 1-form on $M\backslash\overline{D_{1/2}}$.

Now, since $(\alpha, d\rho) = (dH, d\rho)_M = (dH, d\rho)_{D\backslash D_{1/2}} = 0$ for any smooth function compactly supported on $M\backslash\overline{D_{1/2}}$, $\alpha \in E^\perp$, thanks to Proposition 11, α is co-closed. Thus, by 7, α is harmonic in $M \backslash \overline{D_{1/2}}$. $\square$

Claim 3. *p and q are smooth functions on D and, hence, α is smooth.*

Suppose the triple $(\mathcal{U}, \epsilon, \rho)$ satisfies $\mathcal{U} \subseteq D$, *i.e.*, $supp(\rho)$ is inside D.

Thanks to a previously mentioned reasoning, $(dH, d\rho_x) = \iint_D p \cdot \Delta\rho$. Furthermore, $(dH, d\rho_x) = \iint_D (H_x\, dx + H_y\, dy) \wedge (\rho_{xx}\, dx + \rho_{xy}\, dy) = \iint_D H_x \cdot \Delta\rho$. So, subtracting the two expressions, $\iint_D (H_x - p) \cdot \Delta\rho = 0$. Similarly, $\iint_D (H_y - q) \cdot \Delta\rho = 0$, for all smooth functions ρ with compact support in D.

Once again, using Weyl's Lema, both $H_x - p$ and $H_y - q$ are harmonic functions and, in particular, smooth. Therefore, p and q are smooth functions in D.

This means, as we have already proved α is smooth in $M \backslash \overline{D_{1/2}}$, α is a smooth differential in the whole space M. $\square$

Claim 4. *From α and dH we shall extract the function u from the Theorem.*

As $\alpha \in E \subseteq E^{\star\perp}$ and is C^∞, by Proposition 11, α is exact (in M). So let us consider the smooth function f such that $\alpha = df$. Of course, every property α possesses will be maintained to f, in particular, f is a harmonic function in $M \backslash \overline{D_{1/2}}$.

With this new information, remembering the decomposition: $dH = \alpha + \omega$, we can say $\omega = d(H - f) \in E^\perp$.

As inside D, both H and f are smooth, $d(H - f)$ is co-closed in D and of course, it is closed. Hence, $d(H - f)$ is a harmonic differential on D and $H - f$ is harmonic D. Finally, we define on M the mapping:

$$u = f - H + h$$

In $D \backslash \{p\}$, we saw that $f - H$ and h (by definition) are harmonic and on $M \backslash \overline{D_{1/2}}$, f is harmonic and $H - h = 0$. So, u is harmonic on $M \backslash \{p\}$ and we finish the prooof of (i).

Furthermore, we deduce (ii) since $u - z^{-n} = f - H + \overline{z}^n$ and because both $H - f$ and $\overline{z}^n$ are hamonic in D, $u - z^{-n}$ also is in this neighborhood of p.

(iii) comes from the fact that $dH, df \in \mathcal{L}^2(M)$ and $dh = 0$ on $M \backslash D$, therefore, $du = df - dH + dh \in \mathcal{L}^2(M \backslash D)$.

At last, u is harmonic in $M\backslash D$ and $du \in \mathcal{L}^2(M\backslash D)$, so $u \in \mathcal{H}(M\backslash D) = E^{\perp} \cap E^{\star\perp}$, which justifies (iv). ■

3.3 More Properties on Harmonic Functions

Having established the result about the existence of harmonic functions with singularities on any Riemann surface, we are now ready to delve deeper on some of the most essential and intrinsic properties of harmonic functions, namely the Mean-Value Property, Schwarz Reflection Principle and Harnack's Principle, which will be helpful later and of essential knowledge on this topic.

Definition 19. *Consider a continuous function $f : D \to \mathbb{C}$ defined on a domain $D \subseteq \mathbb{C}$. The function f satisfies the Mean-Value property if and only if:*

$$\forall z \in D, \forall r > 0: \quad f(z) = \frac{1}{2\pi r} \cdot \oint_{\partial B(z,r)} f \ .ds = \frac{1}{\pi r^2} \cdot \iint_{B(z,r)} f \ dV$$

Of course, the notation $\partial B(z,r)$ represents the set $\{w \in \mathbb{C} : |w - z| = r\}$. In other words, a function f satisfies the Mean-Value property if it always equals its average value on any closed ball.

Remark. Notice that the double and line integrals in the expression above aren't, in general, equal to each other. However, it can be proven that a function satisfies the Mean-Value Property using the line integral definition if and only if it fulfills the Mean-Value Property using the double integral definition (by using standard techniques of differentiation under integrals and the Fundamental Theorem of Calculus).

Another similar, although seemingly weaker notion is the Local Mean-Value Property (I shall detail this concept on the future chapters):

Definition 20. *Consider a continuous function $f : D \to \mathbb{C}$ defined on a domain $D \subseteq \mathbb{C}$. The function f satisfies the local Mean-Value property if and only if:*

$$\forall\, z \in D,\ \exists\, \delta_z > 0,\ \forall\, r < \delta_z : \qquad f(z) = \frac{1}{2\pi r} \cdot \oint_{\partial B(z,r)} f\ .ds = \frac{1}{\pi r^2} \cdot \iint_{B(z,r)} f\ dV$$

From now on, we will see the beautiful connection between functions which satisfy the Mean-Value Property and harmonic maps:

Theorem. *If u is a harmonic function on a domain region D, then u satisfies the Mean-Value property on D.*

Proof. We use Green's (or Stokes) Theorem and the harmonicity of u for a ball $B(z,r) \subseteq D$:

$$\oint_{\partial B(z,r)} -u_y\, dx + u_x\, dy = \iint_{B(z,r)} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx\, dy = \iint_{B(z,r)} 0\, dx\, dy = 0\,.$$

Also, considering the parametrization $\gamma(t) = z + (r\, cos(t); r\, sin(t))$ of $\partial B(z,r)$:

$$\begin{aligned}
0 &= \oint_{\partial B(z,r)} -u_y\, dx + u_x\, dy = \\
&= \int_0^{2\pi} rsin(t) \cdot \frac{\partial u}{\partial y}\Big(x_0 + rcos(t)\Big) + rcos(t) \cdot \frac{\partial u}{\partial x}\Big(y_0 + rsin(t)\Big)\ dt = \\
&= \int_0^{2\pi} r \cdot \Big[\frac{\partial u}{\partial x} cos(t) + \frac{\partial u}{\partial y} sin(t)\Big]\ dt = \\
&= \int_0^{2\pi} r \cdot \Big(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\Big)\ dt = \\
&= r \cdot \int_0^{2\pi} \frac{\partial u}{\partial r}(z + re^{it})\, dt
\end{aligned}$$

Dividing the above by $2\pi r$ and using the theorem that lets us pull the partial derivatives outside the integral sign, we have $\frac{\partial}{\partial r}\Big(\int_0^{2\pi}\ u(z + re^{it})\ dt \Big) = 0$. Thus,

$$\frac{1}{2\pi} \cdot \int_0^{2\pi}\ u(z + re^{it})\ dt = \text{constant}.$$

In the sense that it does not depend on the variable r, the radius of the ball.

Lemma. *As $r \to 0$, we have* $\left(\frac{1}{2\pi} \cdot \int_0^{2\pi} \; u(z + re^{it})\, dt\right) \longrightarrow u(z)$.

First, notice $\frac{1}{2\pi} \int_0^{2\pi} u(z)\, dt = u(z)$ (it is constant since z is fixed).

Now, let $\epsilon > 0$ and suppose $r < \delta$, where $\delta > 0$ is chosen using (only) the continuity of u at z, such that $w \in B(z, \delta)$ implies $u(w) \in B(u(z), \epsilon)$. Then:

$$\begin{aligned}
&\left| \frac{1}{2\pi} \cdot \int_0^{2\pi} u(z + re^{it})\, dt \; - \; u(z) \right| = \left| \frac{1}{2\pi} \cdot \int_0^{2\pi} \; \left(u(z + re^{it}) - u(z)\right)\, dt \right| \leq \\
&\leq \frac{1}{2\pi} \cdot \int_0^{2\pi} \left| u(z + re^{it}) - u(z) \right| \, dt \leq \epsilon.
\end{aligned}$$

□

Therefore, knowing that $\frac{1}{2\pi} \cdot \int_0^{2\pi} \; u(z + re^{it})\, dt = \textit{constant}$, from this lemma we conclude $\frac{1}{2\pi} \cdot \int_0^{2\pi} \; u(z + re^{it})\, dt = u(z)$.

■

Another useful result concerning harmonic functions consists on their extension under certain restrictions. For example, if we consider a harmonic function defined on the interior of an annulus, is it possible to extend the harmonicity of this function to its border?

Generally, the answer to this question is unfortunately no, as section 5 will show. However, a theorem known as Schwarz's Reflection Principle shall give us sufficient conditions in order to make this extension possible.

Theorem 13 (Schwarz's Reflection Principle)**.** *Let $G \subseteq \mathbb{C}$ be a complex domain for which $\partial G = L$, where L is a circle. Suppose further that u is a harmonic function inside G, continuous on $G \cup L$ and such that $u \equiv 0$ in L. Then, u can be extended to a harmonic function on $G \cup L \cup G^*$, where G^* is the domain symmetric with G relative to L.*

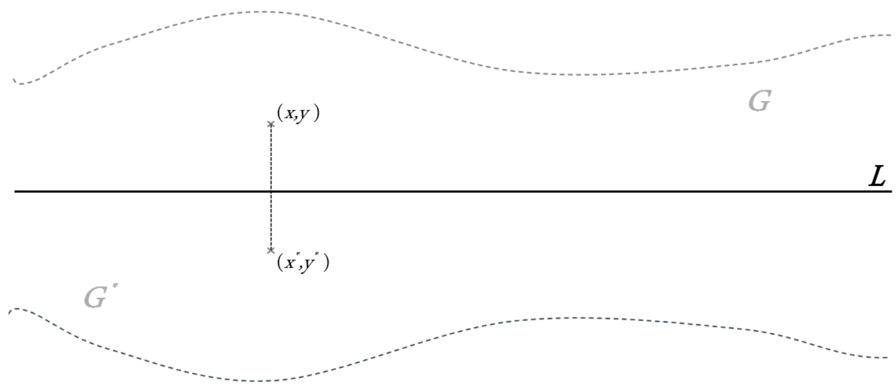


Figure 3: Schwarz Reflection Principle on a domain G inside the upper-half plane.

Remark. Recall that the inversion in, for example, the unit circle is given by $z \mapsto w$ where: $w = \frac{1}{\bar{z}}$.

Proof. As the image above suggests, let's think initially about the case where $G = \mathbb{H}$, the upper half plane and $L = \partial\mathbb{H} = \{z \in \mathbb{C} : \mathfrak{Im}(z) = 0\}$, consider the extension:

$$u(x, y) = u(x, y), \text{ if } (x, y) \in G.$$

$$u(x^*, y^*) = -u(x, y), \text{ if } (x^*, y^*) \in G^* \text{ and } (x^*, y^*) \text{ are symmetric to } (x, y) \text{ with relation to } L.$$

This extension is clearly continuous since it vanishes on L.

All that's left to prove is the harmonicity of u in the whole set $G \cup L \cup G^*$. Unfortunately, the remainder of this statement's proof must be postponed for section 5 where it is learned that the Mean-Value property actually characterizes harmonic functions.

∎

4 Meromorphic Functions on Riemann Surfaces

Meromorphic functions are a generalization of holomorphic maps. In fact, as we know, a meromorphic function on the complex plane is a holomorphic mapping with an isolated set of pole singularities. We shall now study the behavior of these functions on the new context of Riemann surfaces.

4.1 Meromorphic differentials

The assertion of the existence of harmonic functions with singularities is an interesting result in itself, nonetheless, this was mostly a build up to a much more recognizable and important statement about Riemann surfaces: the existence of non-constant meromorphic functions.

Along this chapter, and through the previously mentioned result, our main objective will be to establish this existence for meromorphic 1-forms and later apply it to functions in a simple yet almost magical way.

Definition 21. *A 1-form ω in a Riemann surface M is said to be meromorphic if, locally, $\omega = df$ for some meromorphic function f. More precisely, a meromorphic differential ω is a collection of 1-forms $\{\omega_\phi\}$ for every local coordinate (U, ϕ) in M such that: $\omega_\phi = f_\phi(z)\,dz$ with f_ϕ a meromorphic function defined in U, and such that these are invariant through coordinate changes (as in (9)).*

Theorem 14. *Let M be a Riemann surface, p a point on M and z a local coordinate vanishing at p. Then, for all $n \in \mathbb{N}$, there exists a meromorphic differential on M with a single singularity at p of the type $1/z^{n+1}$.*

Proof. By Theorem 3.2, we can construct a harmonic function u defined on all of M, with a single singularity of type $1/z^n$ at p. Now we consider the 1-form $\alpha = du$, which is a harmonic differential.

We wish to extract a meromorphic differential from α. Remembering Proposition 9 and taking the differential $\omega = \frac{-1}{2n}(du + i^\star du)$ we see that it is a holomorphic differential in M except for its singularity at p, therefore, a meromorphic 1-form in M.

Of course, $u - z^{-n}$ being a harmonic differential at p, $du - dz^{-n} + i^\star(du - dz^{-n})$ will be holomorphic, which means $du + i^\star du - dz^{-n} - i^\star dz^{-n} = -2n\omega + nz^{-(n+1)}dz + nz^{-(n+1)}dz = -2n\omega + 2n/z^{n+1}dz = -2n\,(\omega - \frac{1}{z^{n+1}}\,dz)$ is holomorphic near p. In other words, ω is a meromorphic differential with singularity at p of order $n+1$.

■

4.2 Existence of Non-constant Meromorphic Functions

After establishing the existence of meromorphic differentials on Riemann surfaces, we may now show the analogous result for meromorphic functions.

Theorem. *Let M be a Riemann surface and $p \in M$ a point.*
Then, there exists a non-constant meromorphic function with a single singularity at p.

Proof. Once again, by Theorem 3.2, we can construct a non-constant harmonic function u defined on all of M, with a single singularity of type $1/z^n$ at p. Picking up from where we left in the last proof, we had the meromorphic differential $\omega = \frac{-1}{2n}(du + i^\star du)$. Call this meromorphic differential ω_1. Similarly, one may construct another meromorphic differential, ω_2, from a non-constant harmonic function defined on all of M, with a single singularity in a different point q.

Now, given ω_1, ω_2 two meromorphic differentials as mentioned, we can consider their quotient as a meromorphic function, ω_1/ω_2, on M. This is defined, locally, on a chart (U_α, z_α) as: $\omega_1/\omega_2 = \frac{f_\alpha^1\,\cancel{dz_\alpha}}{f_\alpha^2\,\cancel{dz_\alpha}} = f_\alpha^1 \,/\, f_\alpha^2$ (for the meromorphic functions f_α^1, f_α^2 on U_α

associated to ω_1 and ω_2). The natural question to ask is wether this quotient provides a well defined function, *i.e.*, independent of the local coordinate we choose.

Suppose (U_α, z_α) and (U_β, z_β) are local coordinates for which $x \in U_\alpha \cap U_\beta$, for some point $x \in M$. Notice firstly that, since ω_1 is a (meromorphic) differential, then we must have $f^1_\alpha = f^1_\beta \circ \Phi_{\alpha\beta} \cdot \frac{\partial z_\beta}{\partial z_\alpha}$ on $U_\alpha \cap U_\beta$, where $\Phi_{\alpha\beta}$ is the usual coordinate transition.

Having this established, if one analyzes ω_1/ω_2 through the local coordinate (U_α, z_α) we see: $\omega_1/\omega_2 = f^1_\alpha \ / \ f^2_\alpha = \frac{f^1_\beta \circ \Phi_{\alpha\beta} \ \frac{\partial z_\beta}{\partial z_\alpha}}{f^2_\beta \circ \Phi_{\alpha\beta} \ \frac{\partial z_\beta}{\partial z_\alpha}} = \left(f^1_\beta \ / \ f^2_\beta\right) \circ \Phi_{\alpha\beta}$ and ω_1/ω_2 is, in fact, a well-defined function on M. Besides, as the quotient of meromorphic functions is meromorphic, the existence is finally proven.

$\blacksquare$

5 An introduction to the Dirichlet Problem

Delving deeper in the thematic of harmonic functions we find an interesting subject with connections to the resolution of Partial Differential Equations and their respective boundary conditions, the so called Dirichlet Problem. Beyond the appeal already present in the problem itself, its study shall provide us with a richer understanding of this special type of functions and give a huge contribution to an enormity of posterior theorems and results. Let us, then, proceed to its statement:

The Dirichlet Problem. *Consider D any closed set which is also a differentiable 2-manifold inside $\mathbb{C}$, with respective boundary given by ∂D (notice that, in this case, the topological boundary of D coincides with the boundary of D seen as a manifold with boundary, so that there is no ambiguity when writing "∂D").*

Let f be a continuous function defined on ∂D (one may write this as $f \in C(\partial D)$). Is it possible to construct a continuous function F with domain $D \cup \partial D$ such that:

(i) $F\big|_{\partial D} = f$ and

(ii) F is harmonic inside D ?

In other words given a continuous function on a border, can we extend it to its interior while forcing harmonicity?

Remark. Even though the Dirichlet Problem is considered inside an arbitrary closed region $D \subseteq \mathbb{C}$ with boundary ∂D, our main interest shall lie in the case where the region is a (full) circle (other usual cases like the exterior of a (full) circle or an annulus can be solved thanks to the theory of Fourier Analysis).

Therefore, initially, we will analyze the previously mentioned case where D is a (full) circle and its special intricacies:

5.1 An analysis of the Dirichlet problem for the disc $D = \sigma \cdot \mathbb{D}$

Let g be a real-valued harmonic function on a disc $D = \{z \in \mathbb{C} : |z| < \sigma\}$, for some $\sigma > 0$.

Since D is simply connected, g has a *harmonic conjugate* in D, *i.e.*, there exists a holomorphic function in D such that $g = \mathfrak{Re}\, f$. We can also define the imaginary part of f by:

$$\mathfrak{Im}\left(f(z)\right) = \oint_{[0,z]} {}^{\star}dg \quad ,$$

where $[0, z]$ represents the line segment between 0 and z. This function is well-defined thanks to the path-independence originated from the fact that the disc is simply-connected.

Expanding f in its Taylor series around 0 we have

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{ni\theta}$$

for all $0 < r < \sigma$, $\theta \in [0, 2\pi)$. Therefore,

$$g(re^{i\theta}) = \frac{1}{2} \cdot \left(f(re^{i\theta}) + \overline{f(re^{i\theta})}\right) = a_0 + \frac{1}{2} \cdot \left[\ \sum_{n\geq 1} a_n \cdot r^n e^{ni\theta} + \sum_{n \geq 1} \overline{a_n} \cdot r^n e^{-ni\theta}\ \right]$$

By the theory on Fourier Analysis, we know that the coefficients a_n, for $n \geq 0$, equal:

$$a_0 = \frac{1}{2\pi} \cdot \int_0^{2\pi} g(re^{i\theta})\, d\theta,$$

$$a_n = \frac{1}{\pi} \cdot \int_0^{2\pi} g(re^{i\theta}) \cdot r^{-n} e^{-in\theta}\ \ d\theta, \text{ for } n \geq 1\,.$$

Hence, for $|z| < r$ we have the following expression to f

$$\begin{aligned}
f(z) &= a_0 + \sum_{n=1}^{\infty} a_n z^n = \\
&= \frac{1}{2\pi} \cdot \int_0^{2\pi} g(re^{i\theta}) d\theta + \frac{1}{\pi} \cdot \int_0^{2\pi} \sum_{n \geq 1} g(re^{i\theta}) \cdot (re^{i\theta})^{-n} \cdot z^n \, d\theta = \\
&= \frac{1}{2\pi} \cdot \int_0^{2\pi} g(re^{i\theta}) \cdot [1 + 2 \cdot \sum_{n \geq 1} z^n \cdot (re^{i\theta})^{-n}] \, d\theta \, .
\end{aligned}$$

Note that, since $|z| < r$, the series $\sum_{n \geq 1} \frac{z^n}{r^n e^{in\theta}} = \sum_{n \geq 1} \left(\frac{z}{re^{i\theta}}\right)^n$ converges to $\frac{re^{i\theta}}{re^{i\theta} - z}$ and the last expression equals to

$$\frac{1}{2\pi} \cdot \int_0^{2\pi} g(re^{i\theta}) \cdot \frac{re^{i\theta} + z}{re^{i\theta} - z} \, d\theta \, . \tag{6}$$

Therefore considering its real part:

$$\begin{aligned}
g(z) &= \frac{1}{2\pi} \cdot \int_0^{2\pi} g(re^{i\theta}) \cdot \mathfrak{Re}\left(\frac{re^{i\theta} + z}{re^{i\theta} - z}\right) d\theta = \\
&= \frac{1}{2\pi} \cdot \int_0^{2\pi} g(re^{i\theta}) \cdot \mathfrak{Re}\left(\frac{(re^{i\theta} + z)(re^{-i\theta} - \overline{z})}{|re^{i\theta} - z|^2}\right) d\theta = \\
&= \frac{1}{2\pi} \cdot \int_0^{2\pi} g(re^{i\theta}) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \, d\theta
\end{aligned}$$

The expression inside the integral is directly related to a concept known as the Poisson kernel, relative to the disc D of radius r (about the origin), denominated by P_r and given by

$$P_r(z) = \mathfrak{Re}(\frac{r + z}{r - z}) = \frac{r^2 - |z|^2}{|r - z|^2} = \frac{r^2 - R^2}{r^2 - 2rRcos(\, arg(z)\,) + R^2}$$

Note this is a function that can also be expressed in terms of the argument of z, $arg(z)$ and $R = |z|$, as was done above. In this case, we write $P_r(R\,,\, arg(z))$. Note also, that expression: $\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}$, is exactly $P_r(R = |z| \,,\, arg(z) - \theta)$.

Before proceeding further, we shall first see some properties on the Poisson kernel:

Proposition 15. *For $0 < r < \sigma$, the Poisson kernel satisfies the following properties:*

(i) $\frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\, d\theta = 1$ *for all z such that $|z| < r$.*

(ii) $\frac{r^2-|z|^2}{|re^{i\theta}-z|^2} > 0$ *for all z such that $|z| < r$ and $\theta \in [0, 2\pi)$.*

(iii) $P_r(R,\theta) < P_r(R,\delta)$ *for all θ satisfying $\delta < |\theta| < \pi$ and*

$$\lim_{R=|z|\to r^-} P_r(R,\theta) = 0$$

uniformly on $\delta < |\theta| < \pi$, for all $\delta > 0$.

Proof. In order to prove (i), notice that $1 + 2 \cdot \sum_{n\geq 1} z^n \cdot (re^{i\theta})^{-n} = \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}$ and if we consider $s_n(\theta) = 1 + 2 \cdot \sum_{k=1}^{n} z^k \cdot (re^{i\theta})^{-k}$, by the Weierstrass Theorem for series, we see that $(s_n)_n$ converges uniformly to $\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}$ for $|z| < r'$, with $r' < r$. Thus, it converges uniformly to $\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}$ for $|z| < r$ (as a function of only θ). Hence, we can switch the integral sign with the sum and the result is:

$$\int_0^{2\pi} \sum_{k\geq 1} z^k \cdot (re^{i\theta})^{-k} d\theta = \sum_{k\geq 1} \int_0^{2\pi} z^k \cdot (re^{i\theta})^{-k} d\theta = 0$$

which implies that $\int_0^{2\pi} \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$.

(ii) : Just notice $\frac{r-|z|}{r+|z|} = \frac{r^2-|z|^2}{(r+|z|)^2} \leq \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}$ because $|re^{i\theta} - z|^2 \leq (r + |z|)^2$, and of course, in $|z| < r$ we have both $r > |z|$ and $r + |z| > 0$.

(iii) : $P_r(R,\theta) = \frac{r^2-R^2}{|r-z|^2}$ and therefore:

$$\frac{\partial}{\partial\theta}\Big[P_r(R,\theta)\Big] = (r^2 - R^2) \cdot \Big(- \frac{2rRsin(\theta)}{|r-z|^4}\Big) < 0\,,$$

in the domain $\theta \in (\delta, \pi]$, so the first part of this statement is demonstrated. From this exact result, we also conclude that $P_r(R,\theta) \leq \frac{r^2-R^2}{r^2+R^2}$, so as $R = |z|$ approaches r^-, we see $lim_{R=|z|\to r^-} P_r(R,\theta) = 0$, uniformly on θ (as the bound does not depend on θ) on $\delta < |\theta| < \pi$.

■

Theorem 16. *The Dirichlet problem for the disc D (of any fixed radius r) admits a solution. Furthermore, the solution is unique. In other words, given a continuous function $f\colon \partial D \longrightarrow \mathbb{R}$, there exists a unique continuous function F on $D \cup \partial D$ such that F is harmonic inside D and $F\big|_{\partial D} = f$.*

Proof. To begin with, assuming that a solution exists, let us prove its uniqueness. Consider then u_1 and u_2, two solutions for the the Dirichlet problem in the same disc. Then $u_1 - u_2 \equiv 0$ in the border ∂D. Hence, by the Maximum and Minimum principle for harmonic functions (even though this is a know fact for harmonic functions, an explicit proof of this theorem is given in section 6 where this result can be easily derived through other properties), as $u_1 - u_2$ attains its maximum and minimum in the border, then $u_1 \equiv u_2$.

Let us now prove that the Dirichlet problem admits a solution on the disc $D(0,r)$ of radius $r > 0$. Consider, first, the function F defined as

$$F(z) = \begin{cases} \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{r^2-|z|^2}{|re^{i\theta}-z|^2} \cdot f(re^{i\theta})\, d\theta\,, & \text{if } |z| < r \\ f(z)\,, & \text{if } |z| = r \end{cases}$$

Claim. F is harmonic inside D:

$$F(z) = \int_0^{2\pi} \mathfrak{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) f(re^{i\theta})\, d\theta = \mathfrak{Re} \int_0^{2\pi} \frac{re^{i\theta}+z}{re^{i\theta}-z} \cdot f(re^{i\theta})\, d\theta$$

And the function $f(z) = \int_0^{2\pi} \frac{re^{i\theta}+z}{re^{i\theta}-z} \cdot f(re^{i\theta})\, d\theta$ is analytic, hence, $F = \mathfrak{Re}\, f$ and F is harmonic inside D. □

In other to conclude that F is continuous in ∂D, we will need help from a technical lemma:

Lemma 2. *Let $\alpha \in [0, 2\pi]$ and $\epsilon > 0$. Then, there exists $\rho \in (0, r)$ and an arc A of the circle ∂D such that for $\tilde{\rho} > \rho$ and any $re^{i\tilde{\theta}} \in A$ it verifies that*

$$|F(\tilde{\rho}e^{i\tilde{\theta}}) - f(re^{i\alpha})| < \epsilon\,.$$

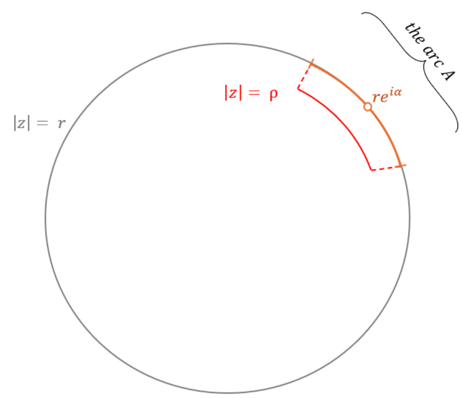


Figure 4: Continuity of F on a neighborhood in ∂D represented by the figure.

Composing with the rotation of angle $-\alpha$, we may assume $\alpha = 0$. Therefore, it suffices to prove the similar statement for $\big|F(\tilde{\rho}e^{i\tilde{\theta}}) - f(r)\big|$. We know for $z = \tilde{\rho}e^{i\tilde{\theta}}$ that

$$\big|F(z) - f(r)\big| = \left|\frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \cdot \big(f(re^{i\theta}) - f(r)\big)\ d\theta\ \right|$$

(thanks to (i) of the last proposition). We now divide the integral above in two pieces: $[0, \delta) \cup (2\pi - \delta, 2\pi]$ and $(\delta, 2\pi - \delta)$, with δ remaining to be chosen.

Initially, through the continuity of f on the border, choose $\delta > 0$ such that if $|\theta| < \delta$ ($\leq 1/2$), then $\big|f(re^{i\theta}) - f(r)\big| < \epsilon/3$. With this estimate, we have $\left(\int_0^{\delta} + \int_{2\pi-\delta}^{2\pi}\right)\ |f(re^{i\theta}) - f(r)| \leq \epsilon/3$, so $\left|\frac{1}{2\pi} \cdot \left(\int_0^{\delta} + \int_{2\pi-\delta}^{2\pi}\right)\ \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \cdot [f(re^{i\theta}) - f(r)]\ d\theta\ \right| \leq \epsilon/3$ (using (i) and (ii)

of the above proposition).

Now, for the second part of the integral, assume M is the maximum of the function f on ∂D, so that $|f(re^{i\theta}) - f(r)| \leq 2M$.

As stated in property (iii) of the previous proposition, there exists ρ such that $\rho \leq |z| < r$ and if $|\eta| > \delta/2$, then $P_r(R\,,\,\eta) \leq \epsilon/(3M)$. Hence, assuming $|\tilde{\theta}| \leq \delta/2$ and because $|\theta| \geq \delta$, just considering $\eta = arg(z) - \theta = \tilde{\theta} - \theta$, we have the second piece of the integral is, in fact, less than or equal to $2/3\,\epsilon$.

Choosing δ satisfying all these conditions, then

$$\begin{aligned}
\left|\frac{1}{2\pi}\cdot\int_0^{2\pi}\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\cdot[f(re^{i\theta})-f(r)]\;\;d\theta\;\right| &\leq \left|\frac{1}{2\pi}\cdot\Big(\int_0^{\delta}+\int_{2\pi-\delta}^{2\pi}\Big)\;\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\cdot[f(re^{i\theta})-f(r)]\;\;d\theta\;\right| \\
&\quad+\frac{1}{2\pi}\cdot\int_{\delta}^{2\pi-\delta}\;\left|\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\cdot[f(re^{i\theta})-f(r)]\right|d\theta \\
&\leq \epsilon/3+\frac{1}{2\pi}\cdot(2\pi-2\delta)\cdot\epsilon/(3M)\cdot 2M \\
&\leq \epsilon\,.
\end{aligned}$$

Thus, the lemma is finally proven and it already implies the continuity of F near the border of the disc. Remembering we we already had its harmonicity inside D, this proof ends.

■

5.2 The Dirichlet problem on the study of Harmonic functions

The existence of a solution for the Dirichlet problem in the circle itself implies important results about harmonic functions.

Theorem. *Suppose f is a continuous function on a domain $D \subseteq \mathbb{C}$ and f satisfies the Mean-Value theorem in the local sense (as in Definition 20) inside D. Then, f is harmonic* *in D.*

Proof. Suppose without loss of generality that f is a function on $\mathbb{R}$. Since f is a continuous function on D, and it is open, choose a disc $D(z_0, r) \subseteq D$. We may now solve the Dirichlet problem on this disc and call that solution u.

Therefore, $u - f \equiv 0$ in $\partial D(z_0, r)$ and $u - f$ satisfies the Mean-Value property locally which in itself implies it fulfills the Minimum and Maximum principle. Let us prove this.

Suppose u satisfies the Local Mean-Value property and that $p \in D$, not in ∂D, is such that $u(p) = m$ is the minimum of u. Then we can find a sufficiently small open ball $B_p(R)$ contained in D so that the (local) Mean-Value property is verified for this ball. Therefore:

$$m = \frac{1}{|B_p(R)|} \cdot \iint_{B_p(R)} u \, dV$$

Now, suppose also q is a point in $B_p(R)$ such that $u(q) > m$ and there is an ϵ small enough for which $u > m$ in $B_q(\epsilon)$. Then,

$$\begin{aligned}
m &= \frac{1}{|B_p(R)|} \cdot \iint_{B_p(R)} u \, dV \\
&= \frac{1}{|B_p(R)|} \cdot \left(\iint_{B_q(\epsilon)} u \ dV + \iint_{B_p(R)/B_q(\epsilon)} u \ dV \right) \\
&> \frac{1}{|B_p(R)|} \cdot m \cdot |B_q(\epsilon)| + \frac{1}{|B_p(R)|} \cdot \iint_{B_p(R)/B_q(\epsilon)} u \ dV \\
&\geq \frac{\epsilon^2}{R^2} \cdot m + m \cdot \frac{\pi(R^2 - \epsilon^2)}{\pi R^2} \\
&= m \, ,
\end{aligned}$$

which is clearly a contradiction. Hence, u in fact satisfies the Maximum (and similarly Minimum) principle.

Returning back to the original proof, as $u - f \equiv 0$ in $\partial D(z_0, r)$, by the Maximum and Minimum principle, $u \equiv f$ and f will be harmonic inside $D(z_0, r)$. As this is true for every point in D, we have global harmonicity in the domain D.

■

Another relevant result that can be proven through the theory on the Dirichlet problem is related to the uniform convergence of harmonic functions.

Corollary 1. *Suppose that M is a Riemann surface. If $(u_n)_n$ is a sequence of harmonic functions on M such that $(u_n)_n$ converges uniformly to a function u in compact subsets of M, then u is harmonic.*

Proof. It is a known fact that the uniform limit of continuous functions is also continuous, hence, u is continuous. Besides this, by the last theorem it suffices to show u satisfies the Mean-Value theorem. This is simple since the uniform convergence allows us to pull the integral sign inside the limit and vice-versa.

■

Now that the foundations of the concept of harmonic functions have been established, we will start a series of some of the most important theorems about these functions, including Harnack's Principle, the fact that the Identity Theorem is maintained in the harmonic case and ending with the conclusion of Schawrz Reflection Principle.

Theorem 17 (Harnack's Inequality). *Let D be a domain in $\mathbb{C}$. Suppose $\tilde{D}$ is a subdomain of D, $\tilde{D} \Subset D$ (i.e., $\tilde{D}$ is relatively compact in D) and u is a positive harmonic function on D. Then, there exists a constant $c = c(D, \tilde{D}) \in \mathbb{R}^+$, only dependent on the domains $\tilde{D}$ and D (it does not depend on the harmonic function we choose) such that*

$$\frac{1}{c} \leq \frac{u(z)}{u(w)} \leq c\,,$$

for all $z, w \in \tilde{D}$.

Remark. Notice, that this statement is equivalent to proving $sup_{\tilde{D}}\, u \leq c \cdot inf_{\tilde{D}}\, u$, for any subdomain in D, with $\tilde{D} \Subset D$ and, of course, $c = c(D, \tilde{D}) \in \mathbb{R}^+$ is independent of the function u above.

Proof. Let $z, w \in \tilde{D}$. There exists a finite chain $\{z, z_1, \cdots, z_k = w\}$ such that each segment $[z_i, z_{i+1}], i \in \{1, \cdots, k\}$ has length $\leq r$.

By the Mean-Value theorem, we have that:

$$u(z) = \frac{1}{|B_z(2r)|} \iint_{B_z(2r)} u\, dV \geq \frac{1}{|B_z(2r)|} \iint_{B_{z_1}(r)} u\, dV = \frac{|B_{z_1}(r)|}{|B_z(2r)|} u(z_1) = \frac{u(z_1)}{4}$$

Therefore, applying this same reasoning to z_1, then z_2, until we get to the end-point w the result is that $u(z) \geq u(w)/2^k$, for some $k \in \mathbb{N}$.

■

Theorem (Harnack's Principle)**.** *Let M be a Riemann surface. Suppose $(u_n)_n$ is a sequence of real harmonic functions, each one defined on a domain D_n of M.*

Assume for each $p \in M$, there is a neighborhood U so that $U \subseteq D_n$, for all $n \in \mathbb{N}$ (except possibly for finitely many) and that

$$u_n(p) \leq u_{n+1}(p)$$

for all $p \in U$ and for all n sufficiently large. Then

(i) either $lim_n\, u_n = \infty$ uniformly for compact subsets on M

(ii) or $lim_n\, u_n = u$ uniformly for compact subsets on M and u is harmonic (by Corollary 1).

Proof. Assume, without loss of generality, that $u_n \geq 0$ (just consider $\tilde{u}_n = u_{n+1} - u_n$), so that, later we are able to apply Harnack's Inequality.

Define the function $u = lim_n \, u_n = sup_n \, u_n$. If we consider compact subsets of M, through Dini's theorem, the convergence is uniform on this compact subsets.

Define also the sets:

$$A = \{x \in M : u(x) < +\infty\}$$

and

$$B = \{x \in M : u(x) = +\infty\}.$$

Just from continuity, we see that A is an open set of M. Through Harnack's inequality, we also see that B is open. Hence, by the connectedness of M, either A or B is empty.

■

Theorem 18 (Identity Principle for Harmonic Functions)**.** *Suppose u is a real harmonic function defined on a domain in the complex plane, D, and let $U \subseteq D$ be a non-empty open set.*

If $u \equiv 0$ on U, then u is identically zero on the whole domain D.

Proof. Write the function $F = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$. Thanks to the Cauchy-Riemann Equations, we can easily see that F is a holomorphic function. As u is zero on the open set U, F also is. Hence, thanks to the (usual) Identity Principle, F vanishes throughout D, as well as u_x and u_y.

Thus, h is constant in D and because it evaluates to 0 on U, $u \equiv 0$ all over D.

■

Before finalizing our study of the Dirichlet problem, the reader should be reminded, in this dissertation, we still have a pendent matter which will now be finally realized: the proof of the Schwarz's Reflection Principle, Theorem 13.

Proof of Schwarz′s Reflection Principle. Let us now demonstrate the Schwarz's Reflection Principle initially for the upper half plane $\mathbb{H}$:

Claim. Let $G \subseteq \mathbb{C}$ be a complex domain for which $\partial G = I$, where I is the real-axis: $\{z \in \mathbb{C} : \mathfrak{Im}(z) = 0\}$.

Suppose u is a harmonic function on G, continuous in $G \cup I$ and $u \equiv 0$ in I.

Then, u can be extended to a harmonic function on $G \cup I \cup G^*$, where G^* is the domain symmetric with G relative to the axis I.

Consider the proceeding extension of u:

$$u^*(x, y) = u(x, y) \text{ , if } (x, y) \in G$$

$$u^*(x, y) = 0 \text{ , if } (x, y) \in I$$

$$u^*(x, y) = -u(x, -y) \text{ , if } (x, y) \in G^*$$

This extension of u is once again continuous, as u vanishes in I.

Let us now prove that u^* is a harmonic function through the Local Mean-Value Property.

For the first case, suppose that $z \in G$. Then, there exists $\delta_z > 0$ so that $B(z, \delta_z) \subseteq G$ and since the local Mean-Value Property implies harmonicity, u is harmonic in this point and on G.

If $z \in G^*$, since $u^*(z) = u^*(x, y) = -u(x, -y)$, through a similar reasoning, we have the harmonicity of the extension on G^*.

Now, if z lies on the axis I, by hypothesis, $u(z) = 0$. Consider any $r > 0$ such that $B_z(r) \subseteq G \cup I \cup G^*$, then:

$$\iint_{B_z(r)} u^* = \iint_{B_z^+(r)} u^* + \iint_{B_z^-(r)} u^*,$$

where $B_z^+(r)$ represents the upper half of $B_z(r)$ and $B_z^-(r)$ the lower one.

Of course, by construction, this is:

$$\iint_{B_z^+(r)} u(x,y)dV + \iint_{B_z^-(r)} -u(x,-y)dV$$

And both integrals cancel out for the result of the sum to be 0, as expected.

Generalizing for circles:

Without loss of generality, assume the circle is in fact the unit circle.

Consider also the bi-holomorphic map $\varphi : z \longmapsto \frac{z-i}{z+i}$ between $\mathbb{H}$ and $\mathbb{D}$. By hypothesis, u is harmonic in $\mathbb{D}$, therefore, $u \circ \varphi$ is harmonic inside $\mathbb{H}$. By the properties of u, $u \circ \varphi$ can be continuously extended to $\{x + iy \in \mathbb{C} : y \geq 0\}$, vanishing in $\partial\mathbb{H}$. Therefore, applying the Schwarz's Reflection Principle for lines proved above, $u \circ \varphi$ can be extended harmonically to $\mathbb{C}$. Call this extension $\tilde{u}$.

Now visualizing φ as a map from $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ to itself. Since this is bi-holomorphic, its restriction from $\mathbb{C}\backslash\{-i\}$ to $\mathbb{C}\backslash\{1\}$ also is, therefore, $u^*(z) = \tilde{u} \circ \varphi^{-1}$ will be harmonic in $\mathbb{C}\backslash\{-i\}$.

In order to deal with this singular point proceed in the same way instead with the map $z \rightarrow \frac{z-i}{-z-i}$. Therefore, we'll have an extension also harmonic but defined on $\mathbb{C}\backslash\{+i\}$.

By the Identity principle for harmonic functions (applied to the difference between these two extensions) we may conclude these coincide and, in the end, the extension will, in fact, be harmonic on the whole complex plane.

$\blacksquare$

6 Super-Harmonic Functions

As we have seen before, harmonic functions play an essential role for a deeper understanding of Riemann surfaces. Exploring this class even further, we find yet another related concept: the super-harmonic functions. These shall be one of the biggest foundations for our proof of the Uniformization Theorem and will allow us to categorize all simply connected Riemann surfaces in just three different types. However, before proceeding, just as an addendum, as I'll make mostly use of functions with image defined on $\mathbb{R}$, from this chapter onwards we may assume all functions are real valued. Having this cleared, let us now start by the definition of super-harmonic functions.

Definition 22 (*Super-Harmonic Function*)**.** *A continuous function $u : M \to \mathbb{R}$ on a Riemann surface M is super-harmonic if for every domain (i.e., open, connected subset) D in M and for every harmonic function h defined in D such that $u \geq h$, either $u \equiv h$ in D or $u > h$ in D.*

Definition 23 (*Subharmonic Function*)**.** *A continuous function $u : M \to \mathbb{R}$ on a Riemann surface M is subharmonic if for every domain (i.e., open, connected subset) D in M and for every harmonic function h defined in D such that $u \leq h$, either $u \equiv h$ in D or $u < h$ in D.*

In other words, a continuous function $u : M \to \mathbb{R}$ is subharmonic if and only if its symmetric, $-u$, is super-harmonic.

Through the existence (or non-existence) of super-harmonic functions fulfilling certain properties we may categorize every simply connected Riemann surface as elliptic, parabolic or hyperbolic surfaces, in the following way:

Definition 24 (*Elliptic, Parabolic and Hyperbolic Riemann Surfaces*)**.** *Let M be a simply connected Riemann surface.*

M is elliptic if and only if M is compact.

M is parabolic if and only if M is not compact and the only positive super-harmonic functions defined in M are the constants.

M is hyperbolic if and only if (M is not compact and) M possesses a non-constant positive super-harmonic function.

Remark 3. As a matter of curiosity, the complex plane $\mathbb{C}$ is parabolic (this can be easily deduced after finishing the proof of the Uniformization Theorem).

As we have seen before, both holomorphic and harmonic functions satisfy the Minimum and Maximum Principle.

However, one shall verify bellow that subharmonic functions actually fulfill only one of these, known as the Maximum Principle for subharmonic functions:

Proposition 19 (Maximum Principle for Subharmonic Functions)**.** *Let D be an open set on a Riemann surface M that is parabolic and u a continuous function defined on its closure: $\overline{D}$. Suppose, further, that u is subharmonic in D.*

If there exist real constants $m_1, m_2 \in \mathbb{R}$ such that $u \leq m_1$ in ∂D and $u \leq m_2$ in $\overline{D}$, then $u \leq m_1$ on $\overline{D}$.

Proof. First of all, if $m_2 \geq m_1$, as $u \leq m_1$ on $\overline{D}$, $u \leq m_2$ on $\overline{D}$.

For the case $m_2 < m_1$, choose $\epsilon > 0$ so that $m_2 + \epsilon < m_1$ and define:

$$v = max\{u\,;\, m_1 + \epsilon\} - m_2, \text{ in } D$$

$$v = m_1 + \epsilon - m_2, \text{ outside } D$$

v is subharmonic in M and a negative function. As M is parabolic, v must be constant. Therefore, $\max\{u\,;\, m_1+\epsilon\} = m_1+\epsilon$ and so $u \leq m_1+\epsilon$, $\forall\, \epsilon > 0$, which leads us to conclude $u \leq m_1$.

■

Remark 4.

Of course switching from subharmonic functions to super-harmonics, we arrive at a similar result just changing "$\leq$" to "$\geq$" above, the so called Minimum Principle for super-harmonic functions.

Remark 5.

Let M be a parabolic Riemann surface and u a bounded harmonic function on an open set D, that is continuous in $\overline{D}$. We may see that, as u is a bounded subharmonic and super-harmonic function (since it is harmonic), then, on $\overline{D}$: $\inf(u|_{\partial D}) \leq u \leq \sup(u|_{\partial D})$.

Corollary (of Proposition 19)**.** *Let M be a Riemann surface and D a conformal disc. Consider u a subharmonic function in M bounded outside D.*

If there exists some real constant $N \in \mathbb{R}$ such that $u \geq N$ in ∂D, then $u \geq N$ (everywhere in M).

Proof. As u is continuous, u is bounded in $\overline{D}$. So, by the hypothesis and the Minimum Principle for subharmonic functions, we have $u \geq N$ in $\overline{D}$. Also, as $\partial(M \backslash D) = \partial D$, using the same argument and the fact u is bounded outside D, $u \geq N$ in $M \backslash D$.
Thus, $u \geq N$ in M.

■

This is an extremely important and useful theorem that shall be used all throughout this dissertation.

Let us now contemplate other equivalent ways to prove the subharmonicity of a continuous function defined on a Riemann surface.

Theorem 20. *Let u be a continuous function on M.*

Then, u is subharmonic in M if and only if for every domain $D \subseteq M$ and every harmonic function h in M: $u+h$ having a maximum in D implies that $u+h$ is constant.

Proof. For the first implication, let N be a constant such that $u + h \leq N$ and suppose there exists a point $p \in D$ where the value N is achieved. Then $u \leq N - h$ which means, by definition, that $u \equiv N - h$ or $u < N - h$, in D and because the second option would contradict our suppositions, we must have $u + h \equiv N$.

Reciprocally, say $u \leq h$, equivalently, $u - h \leq 0$, with h being a harmonic function in a domain D. Either $u - h$ does not achieve a maximum value equal to 0 and $u - h < 0$ or it does achieve 0 at a point. However, this is the same as saying $u - h < 0$ or $u \equiv h$ (the latter resulting from the Maximum Principle for subharmonic functions).

■

Definition 25. *Let M be a Riemann surface.*

Consider $K \subseteq M$ a conformal disc whose closure $\overline{K}$ is contained in a single coordinate z, such that $z(\overline{K})$ is a closed disc in the complex plane $\mathbb{C}$ with radius greater than 1 and center 0, together with a continuous function u defined in M.

We define a new function, denoted by $u^{(K)}$, satisfying the following properties:

(i) $u^{(K)}$ *is continuous on M.*

(ii) $u^{(K)}|_{M\backslash K} \equiv u|_{M\backslash K}$.

(iii) $u^{(K)}$ *is harmonic inside K.*

For a function u and conformal disc K as above, $u^{(K)}$ is called the harmonization of u relative to K.

Remark. The existence and uniqueness of $u^{(K)}$ is given by the solution to the Dirichlet problem from section 5 and its continuous extension along $M\backslash K$.

Proposition 21. *Let $u \in C(M)$, i.e., a continuous function in M.*

Then, u is subharmonic if and only if, for every conformal disc K on M fulfilling the above conditions: $u \leq u^{(K)}$.

Proof. Suppose for now u is subharmonic. Then $u - u^{(K)}$ vanishes in $M\backslash K$. Using the Maximum Principle for subharmonic functions, we see that because $u \leq u^{(K)}$ in ∂K then $u \leq u^{(K)}$ in K (and on M).

For the converse, consider h a harmonic function in a conformal disc D. We will prove the subharmonicity of u through the last theorem, Theorem 20. Assume $u + h$ achieves its maximum, N, in a point in D. Define $D_N = (u+h)^{-1}\{N\} \cap D$.

By hypothesis, D_N is non-empty and it's closed on D. Our objective is now to prove D_N is an open set (in D) and argue with connectedness to conclude that $D_N = D$.

We shall now construct a smaller conformal disc K_r, inside D, with local coordinate z and around a point $p \in D_N$. For $0 < r < 1$, consider the disc $K_r = z^{-1}\big(\{w \in \mathbb{C} : |w| < r\}\big)$ which is a conformal disk, too. In this conditions, recalling the Mean-Value Property for the harmonic function $u^{(K_r)}$:

$$N = u(p) + h(p) \leq u^{(K_r)}(p) + h(p) = (u^{(K_r)} \circ z^{-1})(0) + (h \circ z^{-1})(0) =$$

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} [\,(u^{(K_r)} + h) \circ z^{-1}\,](re^{i\theta})\, d\theta$$

However, by definition, $u^{(K_r)} \equiv u$ in $\partial K_r = \{|z| = r\}$. Hence, the last expression is equal to:

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} [\,(u + h) \circ z^{-1}\,](re^{i\theta})\, d\theta \leq N$$

As the first and last values are the same, we must have $\frac{1}{2\pi} \cdot \int_0^{2\pi} [\,(u+h) \circ z^{-1}\,](re^{i\theta})\, d\theta = N$, for all $0 < r < 1$. Therefore, D_N is simultaneously open and closed, which means $D_N = D$ and $u + h \equiv N$ in D.

$\blacksquare$

This theorem and its proof have several implications, as we shall see in the next corollaries.

Corollary 2. *Consider $u \in C(M)$.*

Then, u is a harmonic function if and only if u is subharmonic and super-harmonic.

Proof. Just notice that if u is harmonic, then $u = u^{(K)}$ always (by standard arguments using the Minimum and Maximum Principles).

Besides this, since the subharmonicity and super-harmonicity of u imply $u = u^{(K)}$ for any conformal disc K as above, this means that u is harmonic in every conformal disc. Thus, in M.

∎

Corollary 3 (from the proof)**.** *Consider u a continuous function on M.*

Then u is subharmonic if and only if for every conformal disc (centered at a point $p \in M$), it satisfies

$$u(p) \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} u(re^{i\theta})\, d\theta, \text{ for all } r < R(p), \text{ where } R(p) \text{ is any small, positive constant.}$$

In particular, if u_1 and u_2 are subharmonic functions, then the function $max\{u_1, u_2\}$ is also subharmonic as well as any linear combination with positive coefficients. We have analogous results for the super-harmonic case.

Corollary 4. *Suppose D is a domain on a Riemann surface M. Assume $u \in C(\overline{D})$, is subharmonic and positive in D and that u vanishes on the boundary ∂D.*

If u is extended to be zero outside D, then this extension is a subharmonic function on M.

Proof. Let us prove the global subharmonicity of this extension through the previous corollary.

Of course, by hypothesis, u is subharmonic on D. Besides this, if $p \in \partial D$, then $u(p) = 0$ and $\iint_{B_p(r)} u = \iint_{B_p(r)\cap D} u + \iint_{B_p(r)\cap(M\setminus D)} u = \iint_{B_p(r)\cap D} u$ and as $u \geq 0$ inside D, therefore, $u(p) \leq \iint_{B_p(r)} u$.

Finally, for the remaining points, we may find a constant $r > 0$ (dependent on the point) such that on this ball, the function vanishes.

■

6.1 Perron Families of subharmonic functions

A special type of families constituted by subharmonic functions on a Riemann surface M are the so called *Perron families*. These kinds of families shall become a useful source for many future conclusions:

Definition 26 (*Perron Family of Subharmonic Functions*)**.** *Let $\mathfrak{S}$ be a family of subharmonic functions on a Riemann surface M. $\mathfrak{S}$ is a called a Perron family if and only if:*

I. $\mathfrak{S}$ is non-empty.

II. $\forall$ conformal disc K in M, $\forall u \in \mathfrak{S}$, $\exists v \in \mathfrak{S}$: $v|_K$ is harmonic and $v \geq u$.

III. $\forall u_1, u_2 \in \mathfrak{S}$, $\exists v \in \mathfrak{S}$: $v \geq$ max $\{u_1, u_2\}$.

Remark 6. From the definition, considering $\mathfrak{S}$ is a Perron family and using the third and second conditions we see that, for every conformal disc K in M, and $u_1, \cdots, u_n \in \mathfrak{S}$ there is a harmonic function h such that: $v \geq \max\{u_1, \cdots, u_n\} \geq u_1, \cdots, u_n$.

The main interest in these families is a very special property fulfilled by their supreme function, illustrated by Perron's Principle.

Theorem (Perron's Principle)**.** *Let $\mathfrak{S}$ be a Perron family on a Riemann surface M.*

If $s : M \to \mathbb{R}$ is defined as $s(P) := sup_{u \in \mathfrak{S}} \{u(P)\}$, then either $u \equiv \infty$ (u is identically ∞) or u is a harmonic function.

Proof. Cover M with coordinate discs $(D_\alpha)_{\alpha \in \Lambda}$. We will actually verify that this principle only needs to be proved in each of these discs. So, let us now assume we have the theorem guaranteed for discs and proceed as follows (of course in the end I'll make sure to verify this is valid in the second claim).

Claim 1. With these assumptions, if u takes the value ∞ on a single disc, then $u \equiv \infty$ on M.

Let's see why. Since M is connected, we may reach any disc D^* from another disc D with a finite chain of discs, that is, there exists a chain of conformal discs: D, D_2, D_3, $\cdots$, D_n, with $D_n = D^*$ such that $D_k \cap D_{k+1} \neq \emptyset$ ($\forall\ 1 \leq k \leq n$).

Now, we could never have s harmonic in a disk and $s \equiv \infty$ in another. Because, if so, then there would be a natural $N \leq n$ such that s harmonic in D_N and $s \equiv \infty$ in D_{N+1}, a contradiction since they intersect.

Therefore, we could never have the two cases mixed together and if $s \equiv \infty$ in a disk, then $s \equiv \infty$ everywhere. $\square$

So, for the case where s harmonic in a single disk D, if we had already proved Perron's Principle for disks, then s must be harmonic in the whole Riemann surface.

All that suffices now is, therefore, to prove Perron's Principle for a conformal disk (because if we had the theorem for discs, in every disc either u is harmonic or identically ∞ and if it never is ∞, u must always be harmonic).

Claim 2. Perron's Principle is valid inside any conformal disc, K, in M.

As K is a conformal disc we may suppose $K \cong \mathbb{D}$, the open unit disk in $\mathbb{C}$. Therefore, as the latter is separable we may find a sequence $(z_n)_n \subseteq K$ that is dense in K.

Besides, since s is a supreme function, for every point z_n we can construct another sequence of functions $(v_{n,k})_k \in \mathfrak{S}$ such that $(v_{n,k}(z_n))_k \to s(z_n)$.

Finally, we'll need yet another sequence of functions in $\mathfrak{S}$. Define, recursively:

Put v_1 harmonic (in K) such that $v_1 \geq v_{1,1}$ (through *II).* And if $v_1, \cdots, v_n$ ($\in \mathfrak{S}$) are defined, then $v_{n+1} \in \mathfrak{S}$ shall be originated in order to satisfy that $v_{n+1}|_K$ is also harmonic, $v_{n+1} \geq v_n$ and $v_{n+1} \geq v_{a,b}$ for all $a, b \leq n+1$ (Remark 6).

This way, $v_m(z_n) \geq v_{a,b}(z_n)$, for all $a, b \leq m$. Also, $\lim_n v_n(z_k) = sup_n \; v_n(z_k) = s(z_k)$ and notice that $(v_n)_n$ was fabricated in order to be an increasing sequence of functions.

We may say that $s(z_1) < \infty$ and using Harnack's Principle in the sequence $(v_n)_n$, $\lim_{n\to\infty}(v_n)$ will be harmonic in K. The only thing left to verify is:

Claim 3. $\lim_{n\to\infty}(v_n) = s$ (all throughout K):

As $(v_n)_n$ is an increasing pointwise converging sequence to v, we may assume the convergence is maintained on $\overline{K}$ (just consider another bigger disk that contains the original K), which is compact. By Dini's Theorem, $(v_n)_n$ converges uniformly on $\overline{K}$ and of course in K, too.

The uniform limit of a sequence of continuous functions is always continuous, so we conclude $\lim_{n\to\infty}(v_n)$ is also continuous and as $\lim_{n\to\infty}(v_n) = s$ agree on a dense set, $\lim_{n\to\infty}(v_n) = s$.

This ends both Claim 2., 3. and the proof.

∎

Perron's Principle will reveal itself as a primordial instrument in the upcoming studies, namely, in the construction of functions sharing some special properties. It will become a routine exercise during proofs to build a Perron family of subharmonic functions, consider their supreme function and verify it satisfies important properties.

6.2 A Return To The Dirichlet Problem on Riemann Surfaces

Now, after having inspected the Dirichlet Problem for the particular case where the domain is a circle, on the complex plane, the natural generalization is to take a domain D, with boundary ∂D, inside a Riemann surface. We will now start our investigation of the Dirichlet problem in these spaces, initially, with a new definition:

Definition 27 (Barrier at a point)**.** *Let D be a domain with boundary ∂D on a Riemann surface M. Consider also a point $p \in \partial D$.*

A function β is a barrier at p if there exists a neighborhood $\mathcal{N}$ of p so that:

(i) β is a continuous function defined on $\overline{D \cap \mathcal{N}}$.

(ii) $\beta(p) = 0$.

(iii) $\beta(Z) > 0$, $\forall\, Z \in \overline{D \cap \mathcal{N}}$ with $Z \neq p$.

(iv) β is super-harmonic in $(D \cap \mathcal{N})^{\mathrm{o}}$ (the interior of $(D \cap \mathcal{N})$).

Definition 28 (Analytic arc)**.** *Let M be a Riemann surface.*

A set L is called an analytic arc on M if L is the image of a line in $\mathbb{C}$ through an analytic map, i.e., there exists a holomorphic function $f : \mathbb{C} \to M$ and a line $\ell \subseteq \mathbb{C}$ such that $f(\ell) = L$.

Analogously, one may define an analytic ray.

Proposition 1. *Let M be a Riemann surface and D a domain on M with a point p in its boundary ∂D.*

If there exists an analytic ray with beginning point p and which does not intersect $\overline{D}$ again, then the point p admits a barrier.

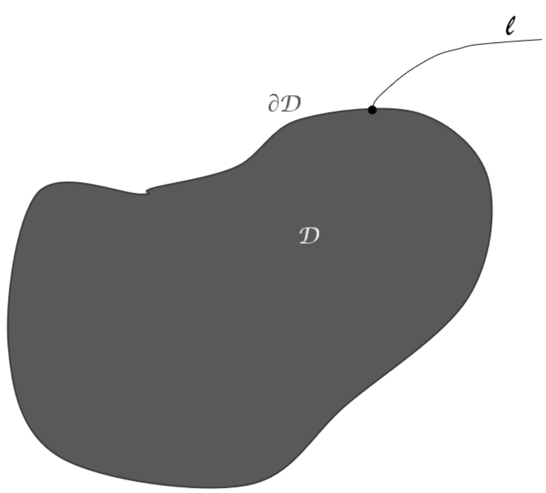


Figure 5: An analytic ray ℓ from a point in ∂D.

Proof. As the existence of a barrier involves only local neighborhoods and results through local coordinates centered at p, we may assume $D \subseteq \mathbb{C}$ and the analytic ray corresponds to the ray $\ell : y = 0, x \leq 0$.

Therefore, it suffices to construct a barrier on $0 \in \partial D$, in this case.

Consider the set $\mathbb{C} \backslash \ell$. This set is simply connected, hence, it has a (holomorphic) square root for the identity function, say $\sqrt{z}$ (for $z \in \mathbb{C} \backslash \ell$).

Now, set $\beta(z) = \mathfrak{Re}\sqrt{z}$ on $\mathbb{C} \backslash \ell$, which writes as $\beta(re^{i\theta}) = r^{1/2} \cdot cos(\theta/2)$, $\theta \in (0, 2\pi]$.

β is harmonic since it is the real part of a holomorphic function, it is positive on all points other than 0 and satisfies (i), (ii), (iii) and (iv). Thus, β is a barrier at p.

∎

From now on, the points $p \in \partial D$ which possess a barrier will be called *regular points*. Furthermore, if all points on the border of D are regular, we say that ∂D *is regular*.

Tying together the concepts of barrier and regular points with the Dirichlet problem, arises a new surprising result which states that the condition of regularity on every point of the border by itself ensures the existence of a solution to the Dirichlet problem on ∂D.

Theorem. *Let M be a Riemann surface, $D \subseteq M$ a domain and a point $p \in \partial D$.*

If every point of ∂D is regular, then, there exists a solution to the Dirichlet problem for every bounded function $f \in C(\partial D)$.

Proof. Let:

$$\mathfrak{F} = \{\, v \in C(\overline{D}) : \; v \text{ is subharmonic in } D,\, inf_{\partial D}\,(f) \leq v \leq sup_{\partial D}\,(f) \text{ and } v\big|_{\partial D} \leq f\big|_{\partial D} \,\}$$

$\mathfrak{F} \neq \emptyset$ because $g \equiv inf_{\partial D}(f)$, the constant function equal to the infimum of f, is on $\mathfrak{F}$.

Besides this, through usual arguments of maximum and harmonization, one may see $\mathfrak{F}$ is a Perron's family of subharmonic functions. Now, define

$$u(Z) = sup_{v \in \mathfrak{F}}\ v(Z),\ z \in D.$$

As $u \leq sup_{\partial D}\ (f)$ on ∂D, u is a harmonic function on D (by Perron's Principle) and $inf_{\partial D}(f) \leq u \leq sup_{\partial D}(f)$.

The only thing left to see that u really is a solution to the Dirichlet Problem is to verify $u\big|_{\partial D} = f$.

We do this through the the following two claims:

[1.] $lim\ inf_{Z \to p}(u(Z)) \geq f(p)$ and

[2.] $lim\ sup_{Z \to p}(u(Z)) \leq f(p),\ for\, all\, p \in \partial D.$

Proof of [1.] :

Firstly, if $f(p) = inf_{\partial D}(f)$, [1.] is proved easily. Henceforth, let's use the notation $inf_{\partial D}(f)= m$ and $sup_{\partial D}(f)= M$.

If not, then there is $\epsilon > 0$ such that $f(p) + \epsilon > inf_{\partial D}(f)= m$. By continuity, there is a neighborhood of p : $\tilde{N}(p)$ where, we have: $f(Z) \geq f(p) - \epsilon$, for $Z \in \tilde{N}(p) \cap \partial D$.

Improving a barrier at a point:

Let us proceed using the hypothesis. So, one may consider a barrier β at $p \in \partial D$ with N the neighborhood of p as in the definition (of barrier) which we may assume is relatively compact. Choose any smaller neighborhood N_0 of p with closure inside N^{o} (the interior of N). Put $m_\beta = min\{\ \beta(X) :\ \mathrm{X} \in \overline{D} \cap \overline{N \backslash N_0}\ \}$ (the set in question is compact so its minimum certainly exists).

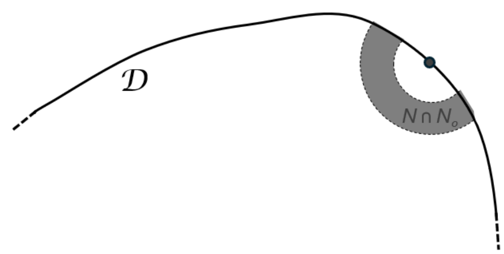


Figure 6: A neighborhood associated to a barrier at a border point.

Then, defining:

$$\tilde{\beta}(Z) = min\{m_\beta, \beta(Z)\},\ Z \in N \cap D\ .$$

$$\tilde{\beta}(Z) = m_\beta,\ Z \in \overline{D \backslash N}\ .$$

$\tilde{\beta}/m_\beta$ is barrier at p which is well-defined on all $\overline{D}$ and equals 1 outside N, called a *normalized barrier at p*. This way, we have just produced an improved version of the usual barrier at a point without using any extra conditions. □

Through this process of "**Improving the barrier at a point**", we may construct a normalized barrier β at p which is 1 outside $N(p)$. After doing so, set:

$$\omega(X) = -(f(p) - m - \epsilon) \cdot \beta(X) + f(p) - \epsilon, \quad X \in \overline{D}.$$

Equivalently:

$$\omega(X) = m \cdot \beta(X) + (f(p) - \epsilon) \cdot (1 - \beta(X)), \quad X \in \overline{D}.$$

Claim 1. $\omega \in \mathfrak{F}$.

Proof of Claim 1.: As $-\beta$ is subharmonic, clearly ω is subharmonic on D.

Besides, β being positive outside p and $f(p) - m - \epsilon > 0$, $\omega \leq f(p) - \epsilon < sup_{\partial D}(f)$ and: $\omega(Z) = m \cdot \beta(X) + (f(p) - \epsilon) \cdot (1 - \beta(X)) > m \cdot \beta(X) + m \cdot (1 - \beta(X)) = m$.

For the third point, suppose $Z \in \partial D$. If Z belongs to ∂D and is outside $\tilde{N}(p)$, because $\beta \equiv 1$, then $\omega(Z) = m \leq f(Z)$ and in case $Z \in \tilde{N}(p) \cap \partial D$, $\omega(Z) \leq f(p) - \epsilon \leq f(Z)$ (remember how $\tilde{N}(p)$ was constructed).

As $\omega\big|_{\partial D} \equiv f\big|_{\partial D}$, $\omega \in \mathfrak{F}$, Claim 1. is, thus, finished. □

Since $\omega \in \mathfrak{F}$, $\omega \leq u$ and by the continuity of ω :

$$lim\ inf_{Z \to p}(u(Z)) \geq lim\ inf_{Z \to p}(\omega(Z)) = \omega(p) = f(p) - \epsilon$$

This remains valid for all $\epsilon > 0$ sufficiently small, so [*1.*] is done. □

Proof of [*2.*]:

Again, let us assume $f(p) < M$ and choose $\epsilon > 0$ so that $f(p) + \epsilon < M$.

Once more, consider also a relatively compact neighborhood $N(p)$ such that $f(Z) \leq f(p) + \epsilon$ on $N(p) \cap \partial D$ and β, as before, a normalized barrier at p relative to $N(p)$.

Claim 2. For all $v \in \mathfrak{F}$, we have:

$$v(Z) - (M - f(p) - \epsilon) \cdot \beta(Z) \leq f(p) - \epsilon, \ \forall Z \in N(p) \cap D.$$

Proof of Claim 2.: Making use of the Minimum Principle for subharmonic functions (which is the case of the function $v - (M - f(p) - \epsilon) \cdot \beta)$ it suffices to prove the claim on $\partial(N(p) \cap D)$.

Because $\partial(A \cap B) \subseteq \partial(A) \cap \partial(B)$, we have two cases for $Z \in \partial(N(p) \cap D)$, case (i): $Z \in \partial N \cap \overline{D}$ and case (ii): $Z \in \overline{N} \cap \partial D$.

In case (i): $Z \in \partial N \cap \overline{D}$, as $\beta \equiv 1$ on ∂N, then $v(Z) - (M - f(p) - \epsilon) \cdot \beta(Z) = v(Z) - M + f(p) + \epsilon \leq f(p) + \epsilon$.

Now, for case (ii): $Z \in \overline{N} \cap \partial D$, as $v \in \mathfrak{F}$, $v(Z) \leq f(Z) \leq f(p) + \epsilon$. □

So, the claim is proven and invoking the supreme

$$u(Z) \leq (M - f(p) - \epsilon) \cdot \beta(Z) + f(p) - \epsilon, \ \forall Z \in N(p) \cap D.$$

Also:

$$lim\ sup_{Z \to p}(u(Z)) \leq lim\ sup_{Z \to p}[(M - f(p) - \epsilon) \cdot \beta(Z) + f(p) - \epsilon] = f(p) - \epsilon$$

Hence, by continuity of u, $u\big|_{\partial D} \equiv f$ and this finishes the proof.

■

Notice this only gives a sufficient condition under which the Dirichlet problem is always solvable for the bounded case.

However, one can actually guarantee an equivalence between regularity and solvability of the (bounded) Dirichlet problem, but initially, let us see what a *proper* solution of the Dirichlet problem is:

Definition 29 (Proper Solution of the Dirichlet Problem)**.** *Let D be a region in M.*

A solution to the Dirichlet problem $u = f$ on ∂D, where $f \in C(\partial D)$ is bounded, is called proper if:

$$inf_{\partial D}(f) \leq u \leq sup_{\partial D}(f) \text{ , on } \overline{D}.$$

Remark. If D is an open set on a Riemann surface M that is parabolic, the solution u is always proper by the Maximum and Minimum Principle for harmonic functions (this, of course includes the cases inside the complex plane).

Using this class of solutions, we may finally arrive to an equivalent condition between regularity on the boundary and solvability of the Dirichlet problem for the bounded case.

Theorem. *Let M be a Riemann surface and $D \subset M$ any set with boudary ∂D. The following are equivalent:*

(i) *For every bounded function $f \in C(\partial D)$, there exists a proper solution for the Dirichlet problem for f.*

(ii) *Every point on ∂D is regular.*

Proof. By the last proof, $(ii) \Longrightarrow (i)$ is already proven.

For the converse, let $p \in \partial D$. We may construct a function f continuous on ∂D with $0 \leq f \leq 1$ and $f(Z) = 0$ if and only if $Z = p$. Consider now the proper solution to the Dirichlet problem for this function f. Since it is proper, by definition, $0 \leq u \leq 1$.

I claim u is strictly positive in D.

This is because if $u(Q) = 0$ for some $Q \in D$, then we may take a small conformal disc around Q and by the Minimum Principle, $u \equiv 0$ on this disc. Using the Identity Theorem, this must occur on the whole Riemann surface. Therefore, as $u > 0$ on D and the point where it equals 0 on ∂D is only p (since there, $u = f$), we conclude $u(X) = 0$ if and only if $X = p$.

Besides, as u is harmonic it is super-harmonic, thus, a barrier at the boundary point p.

■

Consequently, regular points in some nomenclatures can also be called "regular points for the Dirichlet Problem".

6.3 The Harmonic Measure

As mentioned in the end paragraph of the subsection 6.1, the concept of Perron families will become an essential tool used throughout this research. The first time we shall do so, comes in the form of the harmonic measure, a concept I will introduce in a moment, for its only purpose is to work as a middle step to a future purpose.

Definition 30. *Suppose M is a hyperbolic Riemann surface. Let K be a compact subset, with $M \backslash K$ connected and $\partial(M\backslash K)$ regular for the Dirichlet problem.*

Then, there exists a continuous function f on $\overline{M\backslash K}$ satisfying:

(i) f is harmonic in $M\backslash K$.

(ii) $0 < f < 1$ on $M\backslash K$.

(iii) $f = 1$ in $\partial(M\backslash K)$.

Besides, the smallest function satisfying all of these properties (which we still have to prove exists) is called the harmonic measure relative to K: ω_K (or just ω, omitting the compact subset K).

Let us now demonstrate that, on these conditions, there exists a function f on $\overline{M\backslash K}$, satisfying (i), (ii) and (iii).

Proof. By the hypothesis on the Riemann surface, we may consider a function ψ_0 non-constant, positive and super-harmonic on M. Suppose, also, that $m_0 = min_K(\psi_0)$ (we are sure of its existence through the compacity of K) and consider $\psi_1 = \psi_0/m_0$. Therefore, $\psi_1 = \psi_0/m_0$ is once again a non-constant super-harmonic function such that $\psi_1|_K \geq 1$.

By the Minimum principle for super-harmonic functions, we may see that the minimum, 1, of ψ_1 is attained at a point in ∂K. Call this point p_0.

There also exists a point $q \in M\backslash K$ such that $\psi_1(q) < 1$, otherwise, $\psi_1 \geq 1$ on M and this would mean ψ_1 is constant. (Argument as follows: consider a conformal open disc $\tilde{S}$ centered at p_0). Clearly by hypothesis, $\psi_1|_{\tilde{S}} \geq 1$, it is super-harmonic and attains its minimum in the interior of the disc, so by the Identity Theorem for Riemann surfaces, ψ_1 $\equiv 1$.)

Set now $\phi = min\{1, \psi_1\}$ on the whole Riemann surface and note that ϕ is still super-harmonic (it is the minimum of two super-harmonic functions), $0 < \phi \leq 1$, $\phi(q) < 1$ and $\phi|_K \equiv 1$ (because $\psi_1|_K \geq 1$). Remember we are searching for harmonicity, so we aren't quite done yet.

Continuing the proof, define:

$$\mathfrak{F} = \{\ v \in C(\overline{M\backslash K}) :\ v \text{ is subharmonic on } M\backslash K,\ v \leq \phi|_{M\backslash K}\ \}$$

$\mathfrak{F}$ is a Perron's family on $M\backslash K$ since we are authorized to use $v^{(L)}$ (with L a conformal disc) and maxima of two functions in $\mathfrak{F}$ is still in $\mathfrak{F}$. It is also non-empty since $-\phi \in \mathfrak{F}$.

Let us now focus on a particular function that (as we shall verify) belongs on this family and its construction:

Choose a domain D that contains K, with $\overline{D}$ compact and its border ∂D consisting of a finite number of Jordan curves.

Solve the Dirichlet problem on $D\backslash K$ with boundary values equal to 1 on ∂K and 0 on ∂D, denoting the solution v_0 (this is possible since, by hypotheses, ∂K is regular and, by nature, ∂D also is). By the Maximum and Minimum principle for harmonic functions, $0 \leq v_0 \leq 1$ on $\overline{D\backslash K}$. By Corollary 4, v_0 may be extended to a subharmonic function on the set $M\backslash D$, putting $v_0 = 0$ in outside D.

I claim that $v_0 \in \mathfrak{F}$. The only missing statement is that $v_0 \leq \phi$, on $M\backslash K$. Just notice that $v_0 - \phi$ is subharmonic in M and $v_0 \leq \phi$ on ∂K (since both are 1 in this boundary) and the proof of the claim is finished.

Define $\omega(X) = sup_{v \in \mathfrak{F}}\ (v(X)),\ \forall\ X \in \overline{M\backslash K}$.

By construction of the supreme, we know $\omega \leq \phi$. Through Perron's principle, ω will be harmonic in $M\backslash K$.

Also, observe that

$$v_0 \leq \omega \leq \phi.$$

Besides this, since v_0 and ϕ are identically 1 on ∂K, beyond the properties that ω is harmonic on $M\backslash K$, $\omega \equiv 1$ on $\partial(M\backslash K)$ and as $0 \leq \omega \leq 1$ on $\overline{M\backslash K}$, using the fact that non-constant harmonic functions cannot attain their maximum and minimum in the interior, then $0 < \omega < 1$ on $M\backslash K$.

■

Now, in order to obtain the harmonic measure relative to K, consider the slightly modified Perron's family:

$$\tilde{\mathfrak{F}} = \{\ v \in \mathfrak{F} : v \text{ has compact support }\} =$$

$$= \{\ v \in C(\overline{M \backslash K}) :\ v \text{ is subharmonic on } M\backslash K,\ v \leq \phi|_{M\backslash K}, v \text{ has compact support }\}.$$

Define once more $\omega = sup_{v \in \tilde{\mathfrak{F}}}\, v$ on $\overline{M\backslash K}$.

Note that $\tilde{\mathfrak{F}} \neq \emptyset$, because $v_0 \in \tilde{\mathfrak{F}}$ (since it has support in $\overline{D}$). Furthermore:

Claim. ω is the smallest function satisfying properties $(i), (ii)$ and (iii) in the definition, *i.e.*, it is the harmonic measure on K:

Similarly to the argument mentioned before, as we still have $v_0 \leq \omega \leq \phi$, then $\omega = 1$ in ∂K, $0 < \omega < 1$ in $M\backslash K$ and ω is harmonic through Perron's principle.

Suppose, further, that $\tilde{\omega}$ is a function which also fulfills all properties of the theorem.

Let $v \in \tilde{\mathfrak{F}}$ be any element of this family. Using the compact support of v, we may find a compact set K' such that, on $M\backslash K'$, v vanishes.

Furthermore, $\tilde{\omega} \geq 0$ and $v = 0$ on $\partial K'$. Besides, as $\tilde{\omega}|_{\partial(M\backslash K)} = \tilde{\omega}|_{\partial K} \equiv 1$ and $v \leq \phi \leq 1$, we conclude that $\tilde{\omega} \geq v$ on $\partial(K'\backslash K)$. Finally, once again using the Minimum principle for harmonic functions, $\tilde{\omega} \geq v$ on $(K'\backslash K) \cup (M\backslash K'), i.e.,$ on $M\backslash K$.

Taking the supreme over $v \in \tilde{\mathfrak{F}}$, $\omega = \omega_K \leq \tilde{\omega}$. $\square$

Therefore, the harmonic measure ω can always be assembled through this Perron family.

6.4 Green's Functions on Riemann Surfaces

Definition 31 (Green's Function (with singularity at p))**.** *Assume M is a Riemann surface and p a point in M.*

A function g is called a Green's function on M (with singularity at p) if:

I. g is harmonic in $M\backslash\{p\}$.

II. $g > 0$ in $M\backslash\{p\}$.

III. If z is a local coordinate such that $z(p) = 0$, then $g + log|z|$ is harmonic inside a neighborhood of p.

IV. If $\hat{g}$ is another function in M satisfying I., II. and III. then $\hat{g} \geq g$.

Remark 7. Condition *IV* implies the uniqueness of Green's function on M with singularity at p (if we know one already exists).

Remark 8. Condition *III* is invariant under local coordinates at p, meaning that if, for a local coordinate φ we have $g + log|\varphi|$ harmonic in a neighborhood of p, then for any other local coordinate ϕ, $g + log|\phi|$ is also harmonic in a neighborhood of p.

Proof of Remark 8. By definition, "$g + log|\phi|$ is harmonic" means that when composing with a local coordinate it is a harmonic function in the complex plane. Thus, let us choose the local coordinate φ and prove $g \circ \varphi^{-1} + log|\, \phi \circ \varphi^{-1}|$ is harmonic (in $\mathbb{C}$).

As the coordinate change $\phi \circ \varphi^{-1}$ is holomorphic, we can write $\phi \circ \varphi^{-1}(z) = \sum_{n \geq 1} a_n z^n$ with $a_1 \neq 0$ (using a similar reasoning as in Proposition 5) and so $\phi \circ \varphi^{-1}(z) = a_1 z f(z)$, with f holomorphic.

Notice from this we can deduce:

$$g \circ \varphi^{-1}(z) + log|\ \phi \circ \varphi^{-1}(z)| = g \circ \varphi^{-1}(z) + log|a_1| + log|z| + log|f(z)|.$$

By hypothesis, $g \circ \varphi^{-1}(z)$ is harmonic, so all that's left is to see $log|f(z)|$ also is. Assuming our local coordinate is a conformal disc, being simply connected, f possesses a complex (holomorphic) logarithm: $L(z)$, *i.e.*, $f(z) = e^{L(z)}$.
So, $log|f(z)| = log|e^{L(z)}| = \mathfrak{Re}(L(z))$ and finally this is harmonic since real parts of holomorphic functions always are. □

Therefore, the question of uniqueness is now answered by Remark 7. However, for the existence of a Green's function, we must impose that the Riemann surface is of a certain category, more precisely, it must be hyperbolic:

Theorem 22 (Existence of Green's Function)**.** *Let M be a simply connected Riemann surface.*

Then, there exists a Green's function on M (with singularity at some point p) if and only if M is hyperbolic.

Proof. Assume g is a Green's function with singularity at a point p. Set $g_M = \min\{g, M\}$, for $M \in \mathbb{R}^+$.

Since g is a Green's function, g_M will be super-harmonic in $M \backslash \{p\}$ (because if h is harmonic such that $h \leq g_M$, then $h \leq g$ and either $g \equiv h$ or $h < g$ and we had initially $h \leq M$ so $g_M \equiv h$ or $h < g_M$).

Also, as $g(X) \to \infty$ as $X \to p$, near p we have $g_M \equiv M$. Thus g_M is actually super-harmonic in M and as g is positive, $g_M > 0$ in M. The only thing missing is to see g_M is non-constant. If it was, $g \leq M$ and $g - M$ would satisfy *I., II., III.* and $g - M < g$, a contradiction. Therefore, we found a positive super-harmonic non-constant function and the Riemann surface M is hyperbolic.

For the converse assume M is hyperbolic and consider a point $p \in M$ that is covered by a local coordinate (z, K), where K is a conformal disc.

Define the following family of subharmonic functions:

$$\mathfrak{F} = \{v : v \text{ is subharmonic in } M \backslash \{p\}, v \geq 0, \ \text{supp}(v) \text{ is compact and } v + \log|z| \text{ is subharmonic in } K\}$$

Claim. $\mathfrak{F}$ is a Perron's family.

First of all, $\mathfrak{F} \neq \emptyset$ because if we define $v_0 : M \to \mathbb{R}$ as $v_0(P) = -log|z(P)|$ in K and $v_0 \equiv 0$ outside K, $v_0 \in \mathfrak{F}$. And one can see the remaining two conditions to being a Perron's family on $M \backslash \{p\}$ (just use $v^{(K)}$ for *II.* and, for *III.*, consider the maximum).

Before proceeding, we'll need a lemma:

Lemma 3. *$\mathfrak{F}$ is uniformly bounded on the complement of every neighborhood of p.*

Let r be a real constant on $(0, 1)$. Since the Riemann surface M is hyperbolic, we are allowed to consider ω_r the harmonic measure on $\{X \in M : |z| \leq r\}$.

The reader should be reminded that $0 < \omega_r < 1$, $\omega_r = 1$ on $\{|z| = r\}$ and ω_r is harmonic on $\{|z| > r\}$. Let $\lambda_r = max\{\omega_r(z) : |z| = r\}$. Of course, $\lambda_r \in (0, 1)$.

Analogously, write, for $u \in \mathfrak{F}$, $u_r = max\{u(z) : |z| = r\}$.

Claim.

$$u_r \cdot \omega_r - u \geq 0, \text{ for all } |z| \geq r.$$

Proof of claim. The difference $u_r \, \omega_r - u$ is a super-harmonic function on $\{|z| > r\}$ with $u_r \, \omega_r \geq u$ on $\partial(\{|z| \geq r\}) = \{|z| = r\}$ ($\omega_r = 1$ on $\{|z| = r\}$).

Also, as u has compact support and $0 < \omega_r < 1$, the module $|u_r \cdot \omega_r - u|$ is bounded on $\{|z| \geq r\}$, by the Minimum principle for super-harmonic functions, we conclude the above mentioned claim.

Proceeding, remember $u + log\,|z|$ is subharmonic on $|z| < 1$ and continuous on $|z| \leq 1$. Hence:

$$u_r + log\, r \leq max_{|z|=r}\, u + log\, r \leq max_{|z|=1}\, u \leq u_r \cdot \lambda_r$$

Therefore $u_r \leq \frac{-log\ r}{1-\lambda_r}$ and by the compact support of u and the Maximum principle: $max\{u(z) : |z| \geq r\} \leq \frac{-log\ r}{1-\lambda_r}$. $\square$

Continuing our original proof:

Define $s(X) := sup_{u \in \mathfrak{F}}\ \{u(X)\}$, for $X \in M \backslash \{p\}$. As the previous lemma showed, $u < \infty$ for points outside p, therefore, since this is a Perron's family on $M \backslash \{p\}$, by Perron's Principle s is a harmonic function outside p.

This will be our Green's function: $s > 0$ on $M \backslash \{p\}$ because all $u \in \mathfrak{F}$ are. Also, s is harmonic in $M \backslash \{p\}$. Besides this, for *III* consider the local coordinate (z, K) and let us see $s(z) + log|z|$ is harmonic in $\{X \in M : |z(X)| < r\} = K_r$ $(r < 1)$.

Notice, for every $u \in \mathfrak{F}$ (by the Maximum principle): $u(z) - v_0(z) = u(z) + log|z| \leq u_r + log\, r \leq \frac{log(r)}{\lambda_r - 1} + log(r) = \frac{\lambda_r}{\lambda_r - 1} \cdot log(r)$.

Considering the supreme over all $u \in \mathfrak{F}$ we get: $s(z) + log|z| \leq \frac{\lambda_r}{\lambda_r - 1} \cdot log(r)$. Thus, we must have that $s + log|z|$ is harmonic in K_r, thanks to the Removable Singularity Theorem for harmonic functions:

Theorem (Removable Singularity Theorem for harmonic functions)**.** *Suppose u is harmonic on the punctured disc $\{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{D} \backslash \{0\}$, continuous and bounded on $\{z \in \mathbb{C} : 0 < |z| \leq 1\}$.*

Then, u extends to a harmonic function on the whole disc $\mathbb{D}$.

Proof of Theorem. Thanks to the boundedness of $u\big|_{\partial\mathbb{D}}$, we may find a solution v to the Dirichlet Problem on the disc such that $v \equiv u$ on $\partial\mathbb{D}$ (remember v is defined even on the origin).

Define $L_\epsilon(z) = v - u + \epsilon\ log|z| - \epsilon$. Notice $L_\epsilon < 0$ on $\partial\mathbb{D}$, so $L_\epsilon < 0$ on $\mathbb{D}\backslash\{0\}$. Hence, letting ϵ go to 0, we conclude $v \leq u$ on $\mathbb{D}\backslash\{0\}$. Similarly, considering $\tilde{L}_\epsilon(z) = v - u - \epsilon\ log|z| + \epsilon$, we come to the realization that $v \geq u$ on $\mathbb{D}\backslash\{0\}$.

Thus, we can harmonically extend u to the whole disc $\mathbb{D}$ just by putting $u(0) = v(0)$. □

For the finale (*IV.*), let $\hat{g}$ be a competing function, satisfying *I.*, *II.* and *III.*

Lemma 4. *$\hat{g} \geq u, \forall u \in \mathfrak{F}$ by the Minimum principle for super-hamonic functions.*

Put $u \in \mathfrak{F}$. Notice $-u$ is super-harmonic in $M\backslash\{p\}$ and $\hat{g}$ being harmonic, in particular, is super-harmonic in $M\backslash\{p\}$. To cover the point p, just use a similar argument for $\hat{g}+log|z|$ and $u + log|z|$ near p to conclude $\hat{g} - u$ is super-harmonic in all of M.

Remember u has a compact support, say $\tilde{K}$, therefore, $(\hat{g} - u)\big|_{M\backslash\tilde{K}} = \hat{g}\,\big|_{M\backslash\tilde{K}} \geq 0$ and by continuity, $(\hat{g} - u)\,\big|_{(M\backslash\tilde{K})\cup\partial(M\backslash\tilde{K})} \geq 0$. However, $\partial(M\backslash\tilde{K}) = \partial\tilde{K}$, using the Minimum principle for super-hamonic functions (Proposition 19) we must also have $(\hat{g} - u)\,|_{\tilde{K}} \geq 0$. This way, we have just proved $\hat{g} \geq u, \forall u \in \mathfrak{F}$ and of course, $\hat{g} \geq s$.

■

Corollary 5 (of the proof)**.** *Let M be a Riemann surface.*

Then, there exists a Green's function with singularity at one single point in M if and only if $\forall p \in M$: there exists a Green's function with singularity at p (if and only if M is hyperbolic).

Henceforth, we shall represent the Green's function on a (hyperbolic) Riemann surface M with singularity at some point p, evaluated at Q, by: $g_M(p,Q)$ (or simply by $g(p,Q)$).

Let us now explore some properties of hyperbolic Riemann surfaces and their respective Green's functions: g. Beginning with the relation between g and g_D, the Green's function on an open domain D in M (that satisfies some regularity conditions) and finishing on the important statement about their symmetry.

Lemma 5. *M a hyperbolic Riemann surface.*

Let D be a domain on M with $\overline{D}$ compact, such that ∂D is a boundary regular for the Dirichlet problem and consider $p \in D$ a point.

From this conditions, let us consider the unique harmonic function on D that is a solution to the boundary problem $u = g(p, \cdot)$ in ∂D and denote it by u.

Then, $g_D(p,X) = g(p,X) - u(X)$, $\forall X \in D$.

Proof. Let us verify that $g - u$ satisfies all four conditions to be the Green's function on M with singularity at p, so that, by uniqueness, $g - u$ is g_D.

Now, as $\overline{D}$ is compact, $(g-u)\big|_{\overline{D}}$ is bounded bellow and we have $(g-u)\big|_{\partial D} \geq 0$, so by the Minimum Principle for super-harmonic functions, $g - u \geq 0$ in $D \backslash \{p\}$. Also for *III.*, because g is the Green's function with singularity at p, $g + log|z|$ is harmonic near p, thus, $g - u + log|z|$ is harmonic near p.

The only remaining part is to prove *IV.* For this, it suffices to verify that if $\hat{g}$ is a competitor for the Green's function at p then $\hat{g} \geq g - u - \epsilon$, $\forall \epsilon > 0$.

Remember that $g - u \equiv 0$ in ∂D, so for every point $X \in \partial D$, there exists an open set U_X in $\overline{D}$ such that $X \in U_X$ and $g - u < \epsilon$ in U_X. Put:

$$U = \bigcup_{X \in \partial D} U_X$$

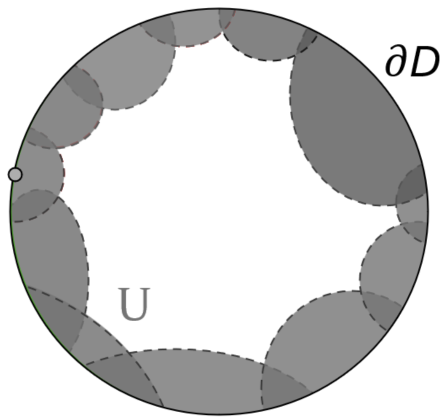


Figure 7: A neighborhood U of ∂D constructed as mentioned in the proof.

Of course, by construction, in U, $g - u < \epsilon$. Thus, $g - u < \epsilon$ in $\partial(D \backslash U) = \partial U$ and using the Minimum Principle for super-harmonic functions, we must have $g - u < \epsilon$ in all of D.

Thus, as > 0, we finally conclude $\hat{g} - (g - u) > 0 - \epsilon = -\epsilon$, *i.e.*, $\hat{g} \geq g - u$.

■

An interesting observation to be made after this proof is that for any domain D in the same conditions as in the above lemma, despite g_D being only defined in the open set D, it can be continuously extended to the whole Riemann surface by declaring $g_D\big|_{M \backslash D} \equiv 0$ (this extension will even be harmonic in $M \backslash \{p\}$ by Corollary 4 since it is subharmonic and super-harmonic).

Lemma 6. *Let D be a domain on a hyperbolic Riemann surface with $\overline{D}$ compact, ∂D consisting of a union of closed analytic arcs and covered by a single local coordinate z centered at p.*

Solve the boundary problem $\gamma = log|z|$ *in* ∂D *and* γ *harmonic in* D*, obtaining a function* $\gamma = u \in C(\overline{D})$*.*

Then, $g_D = u - log|z|$ *is the Green's function on* M *with singularity at* p*.*

Proof. Notice u and $log|z|$ are both harmonic in $D\backslash\{p\}$ and $u - log|z| = 0$ in ∂D, thus, by the Minimum principle for super-harmonic functions, $u - log|z| \geq 0$ in $D\backslash\{p\}$.

Of course, $u - log|z| + log|z| = u$ is harmonic near p by construction. And for $IV.$, it suffices to give a similar argument as in the preceding lemma.

■

Lemma 7. *Suppose* M *is a hyperbolic Riemann surfaces.*

Let D *be a domain on* M *with* $\overline{D}$ *compact,* ∂D *a boundary regular for the Dirichlet problem and* $p, q \in D$ *distinct points in* D*.*

Besides this, assume now that ∂D *is formed by a union of closed analytic arcs (meaning, the image of a line under a holomorphic map).*

Then: $g_D(p, q) = g_D(q, p)$*.*

Proof. Since D is a T_2 space, we may consider open disjoint discs D_1 and D_2 about p and q respectively. Write $\tilde{D} = D\backslash(D_1 \cup D_2)$ and $l(X) = g_D(p, X)$ and $m(X) = g_D(q, X)$.

In this situation, we are authorized to use Stokes Theorem and the result is:

$$0 = \iint_{\tilde{D}} (l\,\Delta m - m\,\Delta l) = \int_{\partial\tilde{D}} (l\,^\star dm - m\,^\star dl) = -[\int_{\partial D_1} + \int_{\partial D_2} (l\,^\star dm - m\,^\star dl)\ ].$$

Now, by the last lemma, we can assume U_1 is a conformal disc with local coordinate z_1 in its polar form $z_1 = re^{i\theta}$ and $l(z) = \tilde{u}(z) - log\, r$, as in the last lemma, with $\tilde{u}(z)$ harmonic in $\{X : |z_1(X)| \leq 1\}$.

Proceeding with calculations (the reader should be reminded of Remark 2. for the third equality and the Mean-Value property of harmonic functions for the following):

$$\int_{\partial D_1} l^\star dm - m^\star dl = \int_{\partial D_1} (\tilde{u} - log\, r)^\star dm - m^\star d(\tilde{u} - log\, r) =$$

$$= \int_{\partial D_1} \tilde{u}^\star dm - m^\star d\tilde{u} - \int_{\partial D_1} (log\, r)^\star dm - m^\star d[log\, r] =$$

$$= \iint_{D_1} \tilde{u}\, \Delta m - m\, \Delta \tilde{u} - \int_{\partial D_1} (log\, r)^\star dm - m(\frac{1}{r} r)\, d\theta =$$

$$= 0 - \int_{\partial D_1} m\ d\theta = -\int_0^{2\pi} m \circ {z_1}^{-1}(e^{i\theta})\ d\theta = 2\pi\ (m \circ z_1^{-1})(0) = 2\pi g_D(q,p)$$

Likewise $\int_{\partial D_2} l^\star dm - m^\star dl = -2\pi g_D(p,q)$. Thus, combining these and using the first equality in the preceding page, we have the desired result.

∎

Theorem (Symmetry of Green's Functions)**.** *Suppose M is a hyperbolic surface. Then:*

$$g(p,q) = g(q,p), \qquad \forall\, p,q \in M.$$

Proof. First of all, consider the family:

$$\mathfrak{D} = \{D \subseteq M \text{ domains} : p \in D, \overline{D} \text{ is compact and } \partial D \text{ is formed by a union of closed analytic arcs}\}.$$

Also define $\mathfrak{F} = \{g_D(p, \cdot) : D \in \mathfrak{D}\}$. As seen in the last observation, we may extend $g_D(p, \cdot)$ to the whole Riemann surface M by claiming it is identically 0 outside D.

Claim. $\mathfrak{F}$ is a Perron's family in $M \backslash \{p\}$.

By Corollary 4, the family is in fact constituted by subharmonic functions.

Consider K a conformal disc in $M \backslash \{p\}$. Choose $D \in \mathfrak{D}$ and a domain $\tilde{D}$ in $\mathfrak{D}$ such that $D \cup K \subseteq \tilde{D}$.

Therefore, $g_{\tilde{D}}$ being harmonic in $M \backslash \{p\}$ implies $g_{\tilde{D}}|_K$ is harmonic and because $g_{\tilde{D}} \geq g_D = 0$ in $\partial \tilde{D}$, once again by the Minimum principle for super-harmonic functions, we'll have $g_{\tilde{D}} \geq g_D$ in the whole closure of $\tilde{D}$.

For the third condition, if $D_1, D_2 \in \mathfrak{D}$, we can find $D \in \mathfrak{D}$ such that $D_1 \cup D_2 \subseteq D$ and this way $g_D \geq g_{D_i}$, $i = 1, 2$. Thus, $\mathfrak{F}$ is a Perron's family in $M \backslash \{p\}$.

By Perron's Principle, $s(X) = sup_{D \in \mathfrak{D}} \{ \, g_D(p, X) \, \}$ will be harmonic in $M \backslash \{p\}$. This is because for all $D \in \mathfrak{D}$ $g_D(p, \cdot) \leq g(p, \cdot)$, thus, $s(X) \leq g(p, X)$. Since s is also a competitor for the Green's function on $M \backslash \{p\}$ (it is positive and harmonic as seen and $\mathfrak{F} + log|z|$ will also be a Perron's family near p, so $s + log|z|$ is harmonic near p), by definition, $s \geq g$. In the end, we conclude $s(X) = g_D(p, X)$.

Hence by Lemma 7., $g_D(p, q) = g_D(q, p) \leq g(q, p)$ and taking the supreme over $D \in \mathfrak{D}$, $g(p, q) \leq g(q, p)$. Switching the roles of p and q, we derive the equality $g(p, q) = g(q, p)$.

∎

This concludes our digression through Green's functions and their properties. Even though, in the future, they will appear again, for Green's functions play an essential role in the proof of the Uniformization theorem for hyperbolic surfaces. It is now time to return to the a before seen theorem: the existence of harmonic functions with singularities on any Riemann surface, and trying to improve this result. Thus, obtaining one of the last steps towards the proof of the Uniformization theorem.

7 Improving the existence of harmonic functions with singularities

Reminding ourselves of the result on the existence of harmonic functions with singularities, as necessary and helpful as this result was (especially for the construction of meromorphic functions on Riemann surfaces) there is but one problem concerning that build up, which can be improved on: the behavior of the harmonic functions with singularity outside small neighborhoods of the singularity point p.

This whole section will be dedicated to building this refinement, focusing on the before mentioned boundedness, in the form of the next theorem:

Theorem. *Consider M a non-hyperbolic Riemann surface and a domain D on M, with a point $p \in D$.*

Let f be a meromorphic function that is holomorphic on $D \backslash \{p\}$.

Then there exists a unique harmonic function u on $M \backslash \{p\}$ that satisfies:

(i) $u - \mathfrak{Re}\, f$ is harmonic in D.

(ii) $(u - \mathfrak{Re}\ f)\,(p) = 0$.

(iii) outside every bounded neighborhood of p: u is bounded.

Remark. The theorem by itself guarantees uniqueness of the function u.

This comes from the fact that if we had u_1 and u_2 fulfilling all conditions, then $u_1 - u_2$ is harmonic on $M \backslash \{p\}$ as well as in D (because $u_1 - \mathfrak{Re}\ f - (u_2 - \mathfrak{Re}\ f) = u_1 - u_2$). Besides, as u is bounded outside every neighborhood, it must be so for one inside D and by continuity of $u_1 - u_2$ on D, $u_1 - u_2$ is bounded everywhere on M. With the help of Liouville's theorem, as $u_1 - u_2$ is harmonic and bounded on M, it must be constant and because $(u_1 - u_2)(p) = 0$ by (ii), then $u_1 - u_2 \equiv 0$.

Let's start with some simple technical lemmas:

Lemma. *Consider a set $D \subseteq \mathbb{C}$, $p \in D$, $u : D \to \mathbb{R}$ a harmonic function and $|u| \leq m$, for some constant $m \in \mathbb{R}^+$.*

Then, there exists a constant C (independent of p) such that:

$$|| \nabla u \, |_p \, ||_{max} \leq \frac{C \cdot m}{d(p, \partial D)}.$$

Remember $|| \cdot ||_{max}$ represents the maximum norm in $\mathbb{R}^2$, that is: $||(a, b)||_{max} = max\{ \, |a|, |b| \, \}$.

Proof. Consider $r > 0$ such that $\overline{B(p, r)} \subseteq D$. As u is harmonic, so is u_x in D. Thus, by the Mean-Value theorem (3.3): $u_x(0) = \frac{1}{\pi r^2} \cdot \iint_{B(0,r)} \, u_x \, dx \, dy$.

Therefore, $|u_x(0)| \leq \frac{1}{\pi r^2} \cdot | \iint_{B(0,r)} \, u_x \, dx \, dy| =$
$= \frac{1}{\pi r^2} \cdot | \int_{-r}^{r} \int_{-\sqrt{r^2 - y^2}}^{\sqrt{r^2 - y^2}} \, u_x \, dx \, dy| =$
$= \frac{1}{\pi r^2} \cdot | \int_{-r}^{r} u(\sqrt{r^2 - y^2}, y) - u(-\sqrt{r^2 - y^2}, y) \, dy| \leq \frac{1}{\pi r^2} \cdot (2r) \cdot (2m) = \frac{4m}{\pi r}$.

■

Lemma 8. *Consider $(u_n)_n$ a sequence of harmonic functions on a domain $D \subseteq \mathbb{C}$.*

If the sequence $(u_n)_n$ converges uniformly on compacts to u, then $(\frac{\partial u_n}{\partial x})_n$ converges uniformly on compacts to $\frac{\partial u}{\partial x}$.

Proof. Notice, first of all, that this result is local in nature. Henceforth, we may assume we want to show the uniform convergence on a simply connected set, say the unit disc.

Therefore, we may choose a holomorphic function f_n such that $u_n = \mathfrak{Re}\, f_n$.

By formula 6 and the theorem which allows one to switch the limit and the integral sign when one has uniform convergence, in fact, f_n converges uniformly to f.

Using again this formula and the theorem which allows one to switch partial derivatives with the integral sign, we also see that, in compact discs: $f\,'_n = (u_n)_x - i\,(u_n)_y$ converges uniformly to $f\,' = u_x - i\,u_y$, hence, $\left(\frac{\partial u_n}{\partial x}\right)_n$ converges uniformly to $\frac{\partial u}{\partial x}$, too.

∎

Lemma 9. *Consider a non-hyperbolic Riemann surface M and a conformal disc, K, in M, as in definition 25. Meaning, a conformal disc whose closure $\overline{K}$ is contained in a single coordinate z, such that $z(\overline{K})$ is a closed disc in the complex plane $\mathbb{C}$ with radius greater than 1 and center 0.*

Assume u is a bounded and harmonic function on $M\backslash K$ and that u is of C^1 class on $\overline{M\backslash K}$. Then:

$$\int_{\partial K} {}^\star du = 0\ .$$

Proof. Assume, firstly, that $u \geq 0$ on $M\backslash K$ (just use a constant that bounds u, say K and put $\tilde{u} = u + K$). As K is a conformal disc, we may consider z its local coordinate, constructed in order to $\delta K = \{\, |z| = 1 + \epsilon \,\}$, for some $\epsilon > 0$.

Through Stoke's theorem and the hypothesis that $\Delta u \equiv 0$ in $M\backslash K$, we have for $\rho > 1 + \epsilon$:

$$0 = \iint_{\{1+\epsilon \le |z| \le \rho\}} \Delta u = \iint_{\{1+\epsilon \le |z| \le \rho\}} d \,{}^\star du = \int_{|z|=\rho} {}^\star du \; - \int_{|z|=1+\epsilon} {}^\star du$$

Thus, $\lim_{\rho \to (1+\epsilon)^+} (\int_{|z|=\rho} {}^\star du) = \int_{|z|=1+\epsilon} {}^\star du$.

Now, in order to simplify the technical language, proceed with a rescaling so that $1+\epsilon$ corresponds to 1. Henceforth, the conclusions we have made become the following: $\lim_{\rho \to 1^+} (\int_{|z|=\rho} {}^\star du) = \int_{|z|=1} {}^\star du$, u is harmonic for $\{|z| > r\}$, for some $r < 1$ and our aim is to prove $\int_{|z|=1} {}^\star du = 0$. Continuing in this vocabulary:

Let $\mathbf{D}$ be the set of domains D such that $K \Subset D \Subset M$ and ∂D is analytic. Consider u_D to be the Dirichlet solution on $\overline{D \backslash K}$ with $u_D\big|_{\partial K} \equiv u\big|_{\partial K}$ and $u_D\big|_{\partial D} \equiv 0$. Extend u_D to be zero outside D.

Define $\mathfrak{F} = \{u_D : D \in \mathbf{D}\}$ and $s = sup_{D \in \mathbf{D}} \{u_D\}$.

In an analogous way to the proof of the Theorem of Symmetry of Green's function, we may verify $\mathfrak{F}$ is a Perron's family on $M \backslash \overline{K}$. Also, because $u \ge 0$ and thanks to another use of the Maximum principle for harmonic functions, we have $\forall \, D \in \mathbf{D}$: $u_D \le u$ outside $\{|z| > 1\}$. Thus, $s \le u$ and similarly, as $u \equiv u_D$ in ∂K, by the same principle $u \ge u_D$, which implies $u \ge s$. Hence $u = s$ in $\{|z| > 1\}$.

In a similar fashion, instead of considering u_D, let ω_D be the solution to the Dirichlet problem on $\overline{D \backslash K}$ harmonic inside but such that $\omega_D|_{\partial K} \equiv 1$ and $\omega_D|_{\partial D} \equiv 0$, still. By the same argument used previously, we see that $sup_{D \in \mathbf{D}} \; \omega_D = 1$ on $|z| > 1$.

Notice that through Schwarz's Reflection Principle, Theorem 13, both u_D and ω_D always possess harmonic extensions across ∂D since both vanish on this border and, also, across ∂K, as we are allowed to consider the differences $u - u_D$ and $\omega_D - 1$ which are harmonic and vanish on ∂K, so all these functions can be harmonically extended beyond $\overline{D \backslash K}$.

Now, through Perron's Principle (and its proof), we may choose sequences of domains $(D_n^1)_n \subseteq \mathbf{D}$ and $(D_n^2)_n \subseteq \mathbf{D}$ such that $lim_n(u_{D_n^1}) = u$ and $lim_n(u_{D_n^2}) = 1$ uniformly on a (compact) neighborhood of ∂K.

Now consider $D_n \in \mathbf{D}$ such that $D_n \supseteq D_n^1 \cup D_n^2$. By construction, $lim_n(u_{D_n}) = u$ and $lim_n(u_{D_n}) = 1$, uniformly on the same before mentioned neighborhood of ∂K. Therefore, we may now proceed with applications of Stokes' Theorem near ∂K:

$$0 = \iint_{D_n \backslash K} (\omega_{D_n} \Delta u_{D_n} - u_{D_n} \Delta \omega_{D_n}) = \int_{\partial(D_n \backslash K)} (\omega_{D_n} {}^\star du_{D_n} - u_{D_n} {}^\star d\omega_{D_n})$$

As $\omega_{D_n} = u_{D_n} = 0$ in ∂D_n, the last expression equals:

$$-\int_{\partial K} (\omega_{D_n} {}^\star du_{D_n} - u_{D_n} {}^\star d\omega_{D_n}) = -\int_{\partial K} (1 \cdot {}^\star du_{D_n} - u \cdot {}^\star d\omega_{D_n})$$

As the convergence is uniform on a neighborhood of ∂K, by Lemma 8, the previous integral converges to $-\int_{\partial K} {}^\star du$ and this integral equals 0.

■

After proving these initial sentences, it is now time to establish the theorem presented in the beginning of this section:

Proof of Theorem 7: Set a local coordinate z that vanishes in p. Assume, without loss of generality, that $D_R = \{X \in M : |z(X)| < R\} \subseteq D$, $R > 1$.

For every $\rho \in (0, 1)$, consider u_ρ the harmonic solution to the Dirichlet problem on the set $M \backslash D_R$ with $u_\rho = \mathfrak{Re}(f)$ in the border $\partial D_R = \{X \in M : |z(X)| = R\}$.

Claim. $\forall\ 0 < r < 1$, $\exists\ c(r) \in \mathbb{R}$, $\forall\ \rho < r$: $\quad |u_\rho(X)| \leq c(r)$ for all $X \in M \backslash D_r$.

Proof of claim. By the Maximum Principle applied to harmonic functions, it suffices to verify the claim in the border $\{|z| = r\}$. Lemma 9 implies that $\int_{|z|=\tilde{r}} {}^\star du_\rho = 0$, for all $\tilde{r}$ with $\rho \leq \tilde{r} \leq R$.

From this, we can define the function F_ρ on $\{\rho < |z| < R\}$ setting: $F_\rho(z) = u_\rho(z_0) + \int_{C_z}(du_\rho + i\ {}^\star du_\rho)$, where C_z is the segment from a point z_0 which also belongs to $\{\rho < |z| < R\}$. F_ρ is holomorphic in $\{\rho < |z| < R\}$ because $du_\rho + i\ {}^\star du_\rho$ represents a holomorphic differential by Proposition 9. Also, remembering u_ρ is real-valued: $\mathfrak{Re}(F_\rho) = u_\rho(z_0) + \int_{C_z} du_\rho = u_\rho(z)$.

Now, as f and F_ρ are holomorphic in the annulus $\{\rho < |z| < R\}$, we may expand their difference in its series: $F_\rho(z) - f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$.

Writing z in its polar form $te^{i\theta}$:

$\mathfrak{Re}(F_\rho(te^{i\theta}) - f(te^{i\theta})) = u_\rho(te^{i\theta}) - \mathfrak{Re} f(te^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n t^n e^{in\theta}$. Now, using the theory around Fourier series, we have the Fourier series expansion of $u_\rho - \mathfrak{Re} f$:

$$u_\rho(te^{i\theta}) - \mathfrak{Re} f(te^{i\theta}) = \frac{\alpha_0}{2} + \sum_{n\geq 1}(\tilde{a}_n^\rho \cdot t^n + \tilde{a}_{-n}^\rho \cdot t^{-n})\ cos(n\theta) + \sum_{n\geq 1}(\tilde{b}_n^\rho \cdot t^n + \tilde{b}_{-n}^\rho \cdot t^{-n})\ sin(n\theta),$$

where $\alpha_0, \tilde{a}_n^\rho, \tilde{a}_{-n}^\rho, \tilde{b}_n^\rho, \tilde{b}_{-n}^\rho$ are constants determined by the following formulas:

$$\alpha_0 = \frac{1}{\pi} \cdot \int_0^{2\pi} u_\rho(te^{i\theta}) - \mathfrak{Re} f(te^{i\theta})\ d\theta.$$

$$\tilde{a}_n^\rho \cdot t^n + \tilde{a}_{-n}^\rho \cdot t^{-n} = \frac{1}{\pi} \cdot \int_0^{2\pi} (u_\rho(te^{i\theta}) - \mathfrak{Re} f(te^{i\theta}))\ cos(nx)\ d\theta.$$

$$\tilde{b}_n^\rho \cdot t^n + \tilde{b}_{-n}^\rho \cdot t^{-n} = \frac{1}{\pi} \cdot \int_0^{2\pi} (u_\rho(te^{i\theta}) - \mathfrak{Re} f(te^{i\theta}))\ sin(nx)\ d\theta, \forall n \in \mathbb{N} \tag{8.1}$$

And, most importantly, for all t in the interval (ρ, R).

Putting $t = \rho$, we obtain the new information that $u_\rho(\rho e^{i\theta}) - \mathfrak{Re} f(\rho e^{i\theta}) = 0$, implying for the above constants: $\alpha_0 = 0$, $\tilde{a}^\rho_{-n} = -\rho^{2n} \cdot \tilde{a}^\rho_n$ and $\tilde{b}^\rho_{-n} = -\rho^{2n} \cdot \tilde{b}^\rho_n$.

Now define $M = max_{|z|=1} \{ \ |f(z)| \ \}$ and $M(\rho) = M_\rho = max_{|z|=1} \{ \ |u_\rho(z)| \ \}$. Using equations (8.1), for $t = 1$, we have the following estimates:

$$|\tilde{a}^\rho_n + \tilde{a}^\rho_{-n}| \leq \frac{1}{\pi} \cdot 2\pi \cdot max_{|z|=1}\{|u_\rho - \mathfrak{Re} f|\} \leq 2(M + M(\rho))$$

and

$$|\tilde{b}^\rho_n + \tilde{b}^\rho_{-n}| \leq 2(M + M(\rho)).$$

Combining these new discoveries with the previous paragraph, then $|\tilde{a}^\rho_n| \cdot |1 - \rho^{2n}| \leq 2(M + M(\rho))$ and $|\tilde{b}^\rho_n| \cdot |1 - \rho^{2n}| \leq 2(M + M(\rho))$.

Thus, for $\rho < 1/2$, $|1 - \rho^{2n}| \geq \frac{1}{2}$ so: $|\tilde{a}^\rho_n| \leq 4(M + M(\rho))$ and $|\tilde{b}^\rho_n| \leq 4(M + M(\rho))$. Of course, in this case, $|\tilde{a}^\rho_{-n}| \leq 4(M + M(\rho))\rho^{2n}$ and $|\tilde{b}^\rho_{-n}| \leq 4(M + M(\rho))\rho^{2n}$ too.

Having established this limits for the constants $\tilde{a}^\rho_n, \tilde{a}^\rho_{-n}, \tilde{b}^\rho_n, \tilde{b}^\rho_{-n}$ and invoking the Fourier series of $u_\rho - \mathfrak{Re} f$:

$$max_{|z|=r} \ |u_\rho| \leq max_{|z|=r} \ |\mathfrak{Re}(f)| \ + \ 8 \sum_{n \geq 1} (M + M_\rho) \cdot (r^n + \frac{\rho^{2n}}{r^n})$$

Noting that, $\rho^2 < r^2$ we may conclude:

$$max_{|z|=r} \ |u_\rho| \leq max_{|z|=r} \ |\mathfrak{Re}(f)| \ + \ 16 \cdot (M + M_\rho) \cdot \frac{r}{1-r}. \qquad (\star\star)$$

However, the reader should be reminded we are in search of a constant $c(r)$, *i.e.*, independent of ρ and the right-hand side of the inequality above does not satisfy this condition. We must, therefore, ensure M_ρ is actually bounded independently of ρ:

By the Maximum principle for harmonic functions, $M_\rho = max_{|z|=1}\ \{\ |f(z)|\ \} \leq max_{|z|=r}\ \{\ |f(z)|\ \}$. Let's say $C(r,\rho) = max_{|z|=r}\ \{\ |f(z)|\ \}$.

Using $(\star\star)$ we see that $M_\rho \leq max_{|z|=r}\ |\mathfrak{Re}(f)|\ +\ 16 \cdot (M + M_\rho) \cdot \frac{r}{1-r}$ hence, choosing r small enough and manipulating this expression, it is easy to prove M_ρ is, in fact, bounded by a constant $c(r)$ only dependent on r. Thus, $|\tilde{a}_n^\rho| \leq c(r)$ and $|\tilde{a}_{-n}^\rho| \leq c(r)\rho^{2n}$, $\forall n \in \mathbb{N}$ and the claim is finally proven. $\square$

Hence, the family of harmonic functions $\{u_\rho : \rho < r\}$ is uniformly bounded in $\{|z| \geq r\}$ (by $c(r)$). Now, consider A the annulus $\{r+K \leq |z| \leq 1\}$, with r small enough as above, $\rho < r$ and $K > 0$ a constant. Clearly, $\{u_\rho : \rho < r\}$ is uniformly bounded in A too.

Notice that we also have equicontinuity of this family in A since if $x, y \in A$ are such that $|x-y| < \delta$, then, using the Mean-Value theorem we have $|u_\rho(x) - u_\rho(y)| \leq ||\nabla u_\rho||_{[x,y]} \cdot |x-y|$ where $||\nabla u_\rho||_{[x,y]} = sup_{P \in [x,y]} ||\nabla u_\rho(P)||$ (and $[x,y]$ is the segment from x to y). But invoking the result in Lemma 7, we have that $||\nabla u_\rho||_{[x,y]} \leq \frac{C \cdot c(r)}{d(p\,,\ \partial(\{|z|>r\}))} \leq \frac{C \cdot c(r)}{K}$.

So using $\delta = \frac{K}{C \cdot c(r)} \cdot \epsilon$: $|u_\rho(x) - u_\rho(y)| \leq \epsilon$, $\forall x, y \in A$. By Arzelà–Ascoli theorem, there is a uniformly convergent subsequence $\{u_{\rho_n} : \rho_n < r\}$ in A. By the Maximum principle for harmonic functions (apply it to the sequence $u_{\rho_n} - lim_n(u_{\rho_n})$), the uniform convergence extends to the set $\{|z| \geq r\}$.

The limit function $U = lim_n(u_{\rho_n})$ is also harmonic in $\{|z| \geq r\}$ by Corollary 1.

Construction of the harmonic function that satisfies $(i), (ii)$ and (iii) through uniformly convergent sequences:

Let's choose a decreasing sequence of radii $(r_k)_k \longrightarrow 0$ and their respective annuluses $A_k = \{r_k < |z| < 1\}$.

Start constructing one sequence $(\tilde{\rho}_m)_m$ in the following way:

From the previous result we may choose, for $r_1 > 0$, a uniformly convergent subsequence $(u_{\rho^1_m})_m$ to U in $\{r_1 < |z| < 1\} = A_1$. Choose $\tilde{\rho}_1$ to be one of these ρ^1_m with $\tilde{\rho}_1 < r_1$ such that $||u_{\tilde{\rho}_1} - U||_{max} \leq 1$ in $A_1 = \tilde{A}_1$.

Having established the term $\tilde{\rho}_N$ we choose $\tilde{\rho}_{N+1}$ in order to belong to a uniformly convergent subsequence $(u_{\rho^N_m})_m$ to U in $\{min\{r_{N+1}, \rho_N\} < |z| < 1\} = \tilde{A}_N$, so that $\tilde{\rho}_{N+1} \leq min\{r_{N+1}, \rho_N\}$ and $||u_{\rho_{\tilde{N+1}}} - U||_{max} \leq 1/(N+1)$ in $\tilde{A}_{N+1}$ ($\supseteq A_{N+1}$).

Thus, we have constructed a sequence $(u_{\tilde{\rho}_n})_n$ that is decreasing, with $\tilde{\rho}_j < r_j$ and $\lim_n (u_{\tilde{\rho}_n}) = U$ in $\{|z| \geq r_j\}$, $\forall j \in \mathbb{N}$, i.e. in $M \backslash \{p\}$. Also, because $U = \lim_n (u_{\rho_n})$ uniformly, it is harmonic in $M \backslash \{p\}$.

In conclusion, U is a harmonic function in $M \backslash \{p\}$ and bounded outside every neighborhood $\{|z| < r_j\}$ with $(r_j)_j \longrightarrow 0$. The only missing detail is to check harmonicity of $U - \mathfrak{Re} f$ in D, (i) and $U(p) = \mathfrak{Re} f(p)$, (ii). Proceed as follows.

Recall that through the Fourier series we demonstrated:

$$(u_\rho - \mathfrak{Re} f)(te^{ix}) = \sum_{n\geq 1} (\tilde{a}^\rho_n \cdot t^n + \tilde{a}^\rho_{-n} \cdot t^{-n})\ cos(nx) + \sum_{n\geq 1} (\tilde{b}^\rho_n \cdot t^n + \tilde{b}^\rho_{-n} \cdot t^{-n})\ sin(nx),$$

and we have the estimates $|\tilde{a}^\rho_n| \leq c(r)$ and $|\tilde{a}^\rho_{-n}| \leq c(r)\rho^{2n}$, $\forall n \in \mathbb{N}$ (similarly for $\tilde{b}^\rho_{-n}$ and $\tilde{b}^\rho_{-n}$).

Putting $\rho \longrightarrow 0$ with the sequence $(\tilde{\rho}_n)_n$, the terms with negative part will vanish and:

$$(U - \mathfrak{Re} f)(te^{ix}) = \sum_{n\geq 1} \alpha_n \cdot t^n\ cos(nx) + \beta_n \cdot t^n\ sin(nx) = \sum_{n\geq 1} (\alpha_n\ cos(nx) + \beta_n\ sin(nx)) \cdot t^n,$$

with $\alpha_n = lim_{\rho\to 0}\ \tilde{a}^\rho_n$ and $\beta_n = lim_{\rho\to 0}\ \tilde{b}^\rho_n$.

This is exactly saying $(U - \mathfrak{Re} f)(z) = \mathfrak{Re} \left(\sum_{n\geq 1} (\alpha_n - i\beta_n) \cdot z^n\right)$, the real part of a holomorphic function, hence, harmonic in the local coordinate z, *i.e.*, near p and because

U is harmonic in $M\backslash\{p\}$ and f holomorphic in $D\backslash\{p\}$, then (i) is valid.

$\blacksquare$

8 The Germs of Holomorphic Functions and Analytic Continuation

The concept of germs on smooth functions is known to be a useful equivalent form of defining the tangent space at a point, particularly fruitful throughout the area of Algebraic Geometry. Having in mind the complex structure given by Riemann surfaces, it is only natural to generalize this concept to the germs of holomorphic functions. After establishing this new perspective, it is only natural to dive deeper into the territory of Algebraic Topology since until this moment, we have not made special use of the simply connectedness condition present in the Uniformization Theorem.

Let us then begin with the exploration of holomorphic/meromorphic germs and proceed by analyzing characters and analytic continuation.

Let M be a Riemann surface.

Definition 32 (Function Element and Germs of Holomorphic Function)**.** *A function element represents a pair $(f,\, \mathcal{U})$, where $\mathcal{U}$ is an open set in M and $f : \mathcal{U} \longrightarrow \mathbb{C}_\infty$ a meromorphic function.*

We may introduce, in the set of function elements on M the following equivalence relation (relative to a point p):

$$f \sim_p g \text{ iff there exists an open set } \mathcal{O} \text{ with } p \in \mathcal{O}, \text{ where: } f|_{\mathcal{O}} \equiv g|_{\mathcal{O}} \,.$$

The equivalence class of the function element $(f,\, \mathcal{U})$ is called the germ of f at the point p. We shall represent this class as $[(f,p)]\,, [f]_p$ or (f,p) depending on the context.

The set of germs of holomorphic functions on a selected point $p \in M$, *i.e.*, the set of equivalence classes of functions at p shall be represented by $\mathcal{O}_p(M)$.

Also, we define:

$$\mathcal{O}(M) = \bigsqcup_{p \in M} \mathcal{O}_p(M).$$

The set $\mathcal{O}(M)$ is called the *sheaf of germs of holomorphic functions* and may be given a topology, as follows.

If $(f, p) \in \mathcal{O}_p(M)$ is a germ, then a neighborhood of this element is:

$N(f,\mathcal{U}) = N_{f,\mathcal{U}}$, the set of germs $\{(f, X) : X \in \mathcal{U}\} = \bigcup_{X \in \mathcal{U}} [(f, X)]$, where $\mathcal{U}$ is an open set containing P. In other words:

$$N(f,\mathcal{U}) = N_{f,\mathcal{U}} = \{\ [g]_X : X \in \mathcal{U} \text{ and } f \sim_X g\ \}$$

Of course, as is usual with this type of construction, we have a noteworthy function, the projection $\pi : \mathcal{O}(M) \to M$, with $\pi([f]_p) = p$.

Another function, well defined from the structure of the germs is the evaluation $e : \mathcal{O}(M) \to \mathbb{C}_\infty$, where $e([f]_p) = f(p)$.

8.1 Characters on a Riemann surface

Consider M a Riemann surface and let $\pi_1(M, p)$ be its fundamental group at a point p.

Definition 33 (Character)**.** *A character χ is a homeomorphism $\chi : \pi_1(M, p) \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, from the fundamental group $\pi_1(M, p)$ onto $\mathbb{C}^*$ (the multiplicative subgroup of the complex plane).*

The set of characters on $\pi_1(M,p)$, considering all points $p \in M$, is denoted by $Char(M)$. A special subset under this category is the one constituted by the normalized characters:

Definition 34. *A character χ is said to be normalized if its range is in the unit circle: $\{z \in \mathbb{C} : |z| = 1\}$.*

Note that, as the abelianization of the fundamental group $\pi_1(M,p)$ corresponds to the first Homology Group, any character can actually also be seen as a homeomorphism with domain $H_1(M,p)$.

8.2 Analytic Continuation along a path

As the term suggests, (direct) analytic continuation/extension consists on the enlargement of the domain under which a holomorphic function is initially defined. More precisely, suppose one has a function g holomorphic in an open set $\mathcal{U}$. If it is possible to find a (bigger) open set $\mathcal{V} \supseteq \mathcal{U}$ and a function G, holomorphic in $\mathcal{V}$ such that $F\big|_{\mathcal{U}} \equiv f$, then F is called the (direct) analytic continuation of f towards the open set $\mathcal{V}$.

An important remark is that the uniqueness implied in the above sentence results from the Identity Theorem.

There exist many processes under which one may extend the holomorphy of a function. We have actually seen a form of this through the Schwarz Reflection Principle (more precisely, its version for holomorphic functions), even though a simpler form of realizing this process, known from Analysis, is through the radius of convergence of power series.

Although, this does not mean that the domain given by the radius of convergence of a power series is the best continuation one may perform. As an example, consider the power series $f(z) = \sum_{k=0}^{\infty}(-1)^k z^k$. Its radius of convergence is only 1, however, one knows this represents the power series for $g(z) = \frac{1}{1-z}$ which is holomorphic on $\mathbb{C}\backslash\{1\}$. This is a holomorphic/analytic extension of f (in fact, the best one is able to construct. As a matter of curiosity, in this case, we call $\mathbb{C}\backslash\{1\}$ the *domain of holomorphy* of f).

Nonetheless, limiting ourselves only to methods of this kind is very restrictive, thus, another useful tool will be introduced in this subsection, the Analytic Continuation along paths:

Definition 35 (Analytic Continuation along a curve)**.** *Consider M a Riemann surface and a curve γ in M, that is, a continuous function $\gamma : I = [0,1] \longrightarrow M$.*

Let f be a meromorphic function on a conformal disc $\mathcal{U}$ centered at $\gamma(0)$, $f : \mathcal{U} \to \mathbb{C}_\infty$.

An analytic continuation of the function element $(f,\mathcal{U})$ along the curve γ is a collection of function elements $(f_t,\mathcal{U}_t)_{t\in I}$ such that :

(i) the initial function element $(f_0,\mathcal{U}_0)$ is $(f,\mathcal{U})$.

(ii) $\forall\ t \in I : \mathcal{U}_t$ is an open disc in M with $\gamma(t) \in \mathcal{U}_t$.

(iii) $\forall\ t \in I$, $\exists\ \delta > 0 : |t - s| < \delta$ implies $\gamma(s) \in \mathcal{U}_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$.

(i.e., one may find an open set containing $\gamma(s)$ where $f_s \equiv f_t$).

The pair $(f_1,\mathcal{U}_1)$ is said to be obtained from $(f,\mathcal{U})$ by (indirect) analytic continuation along the path γ.

Remark. An important remark is that, as any curve in a Riemann surface is compact, one can actually extract just a finite number of open sets $\mathcal{U}, \mathcal{U}_{t_1}, \mathcal{U}_{t_2}, \cdots, \mathcal{U}_{t_N}, \mathcal{U}_1$ (with

$0 < t_1 < t_2 < \cdots < t_N < 1$) which cover the path we are exerting analytic continuation on, hence, holomorphic extension can actually be resumed by a finite collection: $(f,\mathcal{U})$, $(f_{t_1},\mathcal{U}_{t_1})$, $(f_{t_2},\mathcal{U}_{t_2})$, $\cdots$, $(f_{t_N},\mathcal{U}_{t_N})$, $(f_1,\mathcal{U}_1)$.

Remark. The analytic continuation along paths of a function $f : \mathcal{U} \to \mathbb{C}$ is actually a process of indirect analytic continuation, meaning that, for the sequence/chain of open sets described above: $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_N, \mathcal{U}_{N+1}$ one has that $\mathcal{U}_k \cap \mathcal{U}_{k+1}$ and $f_k : \mathcal{U}_k \to \mathbb{C}$ satisfies $f_k \equiv f_{k+1}$ in $\mathcal{U}_k \cap \mathcal{U}_{k+1}$, for all $1 \leq N+1$.

In particular, the analytic continuation (along curves) of a function can depend on the chosen path, even if the paths possess the same beginning and end-points or are actually homotopy equivalent.

This might seem like a contradiction over the uniqueness of analytic extension, however, notice we can actually only establish uniqueness of direct analytic continuation (using the Identity Theorem).

Generally, one only has that analytic continuation along a path γ is unique in the sense that given two analytic continuations along γ, say $(f_t,\mathcal{U}_t)_{t\in I}$ and $(g_t,\mathcal{V}_t)_{t\in I}$, we must have $f_1 \equiv g_1$ on $\mathcal{U}_1 \cap \mathcal{V}_1$ (emphasis on the fact we are taking the same exact path).

To see an example that illustrates this concept, let γ be the closed path in $\mathbb{C}$ given by: $\gamma(t) = e^{2\pi i t}$, $t \in [0,1]$ and the complex Logarithm. This function is holomorphic when defined on a neighborhood of 1, although, when performing analytic continuation along this path we end up at a value which is $2\pi i$ plus the original one.

Theorem 23. *Consider σ an isolated set on a Riemann surface M.*

Let u be a harmonic function on $M\backslash\sigma$.

Assume, also, for every point $p \in M$, there exists a neighborhood $N[p]$ and a meromorphic function f_p defined on $N[p]$ such that, just one of the following conditions is fulfilled:

[1.] $\mathfrak{Re}\, f_p = u$ *on* $N[p]\backslash\sigma$,

Or:

[2.] $log\,|f_p| = u$ *on* $N[p]\backslash\sigma$.

Then, choosing any function f_p *described as above, we have that:*

(i) f_p *can be analytically extended along any curve in* M *with beginning point* p.

(ii) *the continuation of* f_p *along any loop with starting and end-point equal to* p *depends only on the homology class of the loop in question.*

Proof. Consider a continuous path on M, $c : I = [0,1] \to M$, with $c(0) = p$. And write the following set:

$$L = \{\ t \in [0,1] : f_p \text{ has an analytical extension along } c\big|_{[0,t]} \text{ satisfying hypothesis } [\mathit{1.}] \text{ (or } [\mathit{2.}]) \ \}$$

L is a non-empty set of $[0,1]$, for $0 \in L$ since, by the hypothesis, one may consider the function element $(f_p, N[p])$ around $c(0) = p$. L is an open set thanks to the condition (iii) of the of definition analytic extension.

Let us show L is also closed (supposing [*1.*]):

Suppose $(t_n)_n \subseteq L$ is a sequence converging to $\bar{t}$ and demonstrate $\bar{t} \in L$.

For all $N \in \mathbb{N}$, consider $(f_{t_{kN}}, \mathcal{U}_{kN})_k$, $k = 1, \ldots, m$, the function elements that come from the analytic continuation along c to the point $c(t_N)$ (using the compactness of $c_{|_{[0,t_N]}}$). Let us abbreviate this notation to just $(f_{t_k}, \mathcal{U}_k)_k$, $k = 1, \ldots, m$ (even though the reader should be reminded this is a sequence dependent on N).

Consider the function given by the hypothesis: $f_{c(\bar{t})}$ and its respective neighborhood $N[c(\bar{t})]$. There exists some N and some function element $(f_m, \mathcal{U}_m)$ as above such that $\mathcal{U}_m \cap N[c(\bar{t})] \neq \emptyset$. As L imposes that f_m satisfies hypothesis [*1.*], then $\mathfrak{Re}(f_m) = u = \mathfrak{Re}(f_{c(\bar{t})})$. Hence, there exists a constant $C \in \mathbb{R}$ such that $f_m = f_{c(\bar{t})} + i\,C$ on $\mathcal{U}_m \cap N[c(\bar{t})]$. This is exactly an extension of the function along $c\big|_{[0,\bar{t}]}$ and so $\bar{t} \in L$. This finishes the proof of (i) through a usual connectedness argument. The reasoning for hypothesis [*2.*] is very similar.

Now, after establishing that one may analytically extend along any curve in M, let us prove statement (ii).

Suppose c is a closed path with beginning and end-point $p \in M$ and consider φ_1 the analytic continuation of f_p along c. Then, near p, by hypothesis [*1.*], we have $\mathfrak{Re}\, \varphi_1 = u = \mathfrak{Re} f_p$ in a neighborhood of p. Therefore, $\varphi_1 = f_p + i\,\chi(c)$, for some $\chi(c) \in \mathbb{R}$ dependent on c.

If one assumes hypothesis [*2.*], we have $log|f_p| = u = log|\varphi_1|$. This implies that there exists a constant dependent on the path c, say $\chi(c) \in \mathbb{C}$, with $|\chi(c)| = 1$ such that: $\varphi_1 = \chi(c) \cdot f_p$.

Recall the Monodromy Theorem:

Theorem (Monodromy Theorem)**.** *Suppose D is an open domain in a Riemann surface M, $p \in D$ and f is a holomorphic function defined in D.*

If two loops γ_0 and γ_1 beginning at p are homotopic equivalent, through a homotopy γ_s, $s \in [0,1]$ and one is able to perform analytic continuation along every path γ_s, then the analytic extension of f along γ_0 and γ_1 will yield the same result on the end-point p.

Hence, $\chi(c)$, in both cases, depends, actually on the homotopy class of the loop c. Therefore, χ is a character on $\pi_1(M,p)$ and, as we have seen before, $\chi(c)$ depends only on the homology class of c.

■

Remark. Consider the Riemann surface $M = \mathbb{C}$, $\sigma = \{0\}$, the complex Logarithm and the function $u = log\,|z|$ and let γ be the loop $\gamma(t) = e^{2\pi i t}$, $t \in [0,1]$.

Of course, u is harmonic on $\mathbb{C}\backslash\{0\}$. Besides, in every point $p \in \gamma$ (the image of γ), $p \neq 0$, the (usual) complex Logarithm, L, is a holomorphic function such that $\mathfrak{Re}\, L = log\,|\cdot|$.

For the point 0, we may find another branch of the complex Logarithm $\tilde{L}$ such that $\mathfrak{Re}\, \tilde{L} = log\,|\cdot|$ in a neighborhood of 0. Hence, $\mathbb{C}\backslash\{0\}$ and u fulfill the hypothesis *[1.]* and, as a matter of fact, when performing analytic continuation along γ we obtain the original value of the logarithm summed by $2\pi i = i \cdot \chi(\gamma)$.

Theorem 24. *Suppose M is a simply connected Riemann surface.*

Consider σ an isolated set on a Riemann surface M and u be a harmonic function on $M\backslash\sigma$.

If u satisfies either one of the two conditions [1.] (resp. [2.]) locally, as in Theorem 23, then u is globally the real part of a meromorphic function defined on M (resp. the log modulus of a meromorphic function defined on M).

Proof. A simple corollary of last theorem. ■

9 The Uniformization Theorem for Hyperbolic Surfaces

We shall now begin the ending steps on the final proof of the Riemann Uniformization Theorem, starting with the demonstration that every simply connected hyperbolic Riemann surface is conformally equivalent to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem (Uniformization Theorem, Part I)**.** *If M is a simply connected hyperbolic Riemann surface, then M is conformally conformally equivalent to the complex unit disc $\mathbb{D}$.*

Proof. Consider a point $p \in M$ and $g(p, \cdot)$ the Green's function on M with singularity at the point p, whose existence is guaranteed from the fact M is hyperbolic. Notice, also

Claim 1. $g(p, \cdot)$ satisfies hypothesis [*1.*] of Theorem 23.

For every point $Q \neq p$, we may just choose a simply connected neighborhood U_Q where, of course, there exists a meromorphic function f_Q such that $\mathfrak{Re}(f_Q) = g(p, \cdot)$.

For the point p itself, choose a simply connected local coordinate z vanishing at p. The reader should be reminded that $v = g(p, \cdot) + log|z|$ is harmonic near p. In this simply connected local coordinate, we may find an harmonic conjugate v^* so that $h = v + iv^*$ is holomorphic. Besides, define: $f = e^{v+iv^*}$. Thus:

$$v = g(p, \cdot) + log\,|z| = log(e^v) = log\,|f|, \text{ on a neigborhood of } p.$$

Hence:

$$g(p,z) = log\left|\frac{f(z)}{z}\right|, \text{ on a neigborhood of } p. \quad \square$$

Applying Corollary 24 to the symmetric of the Green's function, $-g(p,\cdot)$, since the Riemann surface M is simply connected, there exists a global meromorphic function, say $f(p,\cdot)$ such that: $log\,|f(p,X)| = -g(p,X)$, $\forall\, X \in M$.

We will now prove the function $f(p,\cdot)$ is, fact, a conformal function onto the unit disc, *i.e.*, a bijective and holomorphic function.

$f(p,\cdot)$ **is a holomorphic function onto the unit disc:**

For $X \neq p$, by definition, $g(p,X) > 0$, therefore, $|f(p,X)| = e^{-g(p,X)} < 1$.

Also, through this boundedness and by the Riemann Removable Singularity theorem from Complex Analysis, $f(p,\cdot)$ extends itself to a holomorphic function on the whole Riemann surface M (with $f(p,p) = 0$).

$f(p,\cdot)$ **is injective:**

Consider $X, Y, Z \in M$, where Z is the only variable (both points X and Y are fixed) and the transformation:

$$\varphi(Z) = \frac{f(X,Z) - f(X,Y)}{\overline{f(X,Y)} \cdot f(X,Z) - 1}, \quad \text{for } Z \in M.$$

Notice, firstly, that if $|z_0| < 1$, the Möbius transformation:

$$z \mapsto \frac{z - z_0}{\overline{z_0} z - 1}$$

fixes the open unit disc. This property will be used repeatedly going forwards.

Continuing, φ is a composition of $f(X, \cdot)$ followed by a Möbius transformation that fixes the unit disc, so φ is holomorphic and $\varphi(Y) = 0$ and $|\varphi| < 1$.

Claim 2. There exists $n \in \mathbb{N}$ such that: $|\varphi(Z)|^{1/n} \leq |f(p, Z)|$, $\forall\ Z \in M$.

Set ξ a local coordinate vanishing at p. We can assume, without loss of generality, that this local coordinate is mapped to a neighborhood of the unit closed disc.

Using the Local Normal Form, Proposition 5, of the holomorphic function φ:

$$\varphi \circ \xi^{-1}(z) = az^n(1 + a_1 z + a_2 z^2 + \cdots), \text{ with } n \geq 1 \text{ and } a \neq 0.$$

Define, now: $u(X) = -\frac{1}{n}\ log|\varphi(X)|$.

As u is the logarithm of the modulus of a holomorphic function, it is harmonic and positive, outside the set of zeros of φ.

Now, for the remainder of the proof of this claim, the reader should be familiar with a before seen Theorem: 22, as we shall use many techniques borrowed from there.

Consider the same family of subharmonic functions:

$$\begin{aligned} \mathfrak{F} = \{v : v \,\text{is subharmonic in } M \setminus \{p\}\,, v \geq 0, \\ v \text{ has compact support and } v + log\,|\xi| \text{ is subharmonic near } p\} \end{aligned}$$

Select any function $v \in \mathfrak{F}$. Denote by $D(v) = D$ its (compact) support and consider D_o a set constituted by the union of small discs about the singularities of u and where we suppose $u \geq v$ (we may suppose so since the singularities of u imply that near these points, the function grows immensely).

In case $D \cap D_o = \emptyset$, we shall extend the set D in order to intersect D_o. This can be done in the following way: consider a point on D and a path leading this point towards a singularity in D_o. Cover this path with small open sets on M and instead of proceeding with the set D, we shall use the set $\tilde{D}$ formed by the closure of the union of D, with the before mentioned open sets and the small discs on D_o around the chosen singularities. This reasoning is summarized in the picture:

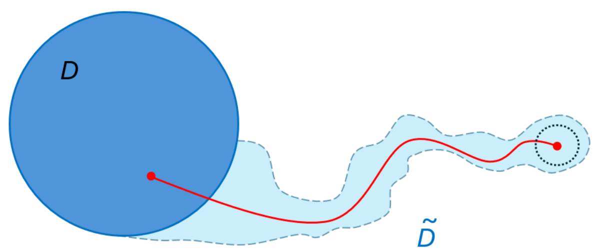


Figure 8: An extension, $\tilde{D}$, of the domain D that contains singularities.

As I have noted earlier, since u has a singularity, in one of its neighborhood $W \subseteq \tilde{D}$, certainly $u \geq v$. Therefore, $0 \geq v - u$ on $\partial W = \partial(\tilde{D} \backslash W)$ and by the Maximum Principle for subharmonic functions, $u \geq v$ in $\tilde{D}$ and, as a consequence, in D. Hence, (as D is the support of v) in all of $M \backslash \{p\}$.

Thus, taking the supreme: $-\frac{1}{n}\ log|\varphi(X)| = u(X) \geq sup_{v \in \mathfrak{F}}\ v(X) = g(p, X) = -log$ $|f(p, X)|$. Manipulating this expression, $|\varphi| \leq |\varphi|^{1/n} \leq |f(p, \cdot)|$ and the claim is proven. $\square$

Continuing the proof, construct, now, the holomorphic map from M to the closed unit disc $\overline{\mathbb{D}}$:

$$h(Z) = \frac{\varphi(Z)}{f(p, Z)}$$

By the above claim, $|h|$ is bounded by 1 and we have that $|h(X)| = 1$ if we choose $Y = p$ (and use the symmetry of Green's function). Invoking the Maximum Modulus Principle, we have that h is constant and so, there exists $\chi \in \mathbb{C}$, $|\chi| = 1$, for which $\varphi(Z) = \chi f(p, Z)$.

Using the expression of $\varphi(Z)$ and choosing $p = Y$:

$$\chi f(p, Z) = \frac{f(X, p) - f(X, Z)}{1 - \overline{f(X, p)} \cdot f(X, Z)}$$

And so, we deduce:

$$f(X, p) = f(X, Z) \Longleftrightarrow f(p, Z) = 0 \Longleftrightarrow Y = Z.$$

That is, $f(p, \cdot)$ is injective.

$f(p, \cdot)$ **is onto the open unit circle:**

Suppose there exists an $a \in \mathbb{D}$ such that $a \notin Im(f(p, \cdot))$. From this point forward, we shall represent $f(p, \cdot)$ as just f, in order to simplify the language.

As $f(p, p) = 0$, we know $a \neq 0$. Besides this, as $f(p, \cdot)$ is an injective holomorphic function and $\pi_1(M) = 0$, by Invariance of Domain, then $f(M)$ is also simply connected. In particular, every non-vanishing holomorphic function on $f(M)$ possesses a square root.

Hence, for example, defining $\omega : z \longmapsto \frac{z-a}{1-\bar{a}z}$, then ω, in fact, has a square root in $f(M)$: $\sqrt{\omega} = \sqrt{\frac{z-a}{1-\bar{a}z}}$ (this transformation has image in the open unit disc).

Consider, also:

$$t(z) = \frac{\omega(z) - i\sqrt{a}}{1 + i\,\overline{\sqrt{a}} \cdot \sqrt{\omega(z)}} = \frac{\sqrt{\frac{z-a}{1-\bar{a}z}} - i\sqrt{a}}{1 + i\,\overline{\sqrt{a}} \cdot \sqrt{\frac{z-a}{1-\bar{a}z}}} \quad , \; z \in f(M).$$

Where we may choose a branch of $\sqrt{\cdot}$ such that $\sqrt{-a} = i \cdot \sqrt{a}$.

Hence, $t(0) = 0$, $|t(z)| < 1$ for $z \in f(M)$ (because t is the composition of ω and another transformation that fixes the open unit disc) and $|t'(0)| > 1$.

However, by Schwarz lemma, as $t(0) = 0$ and $|\,t\,| < 1$, we know that $|t'(0)| < 1$. We arrived at a contradiction.

Thus, there does not exist such $a \in \mathbb{D}$ so that $a \notin Im(f(p,\cdot))$, $f(p,\cdot)$ is also surjective and a conformal map.

■

Thus, we have finally established the Uniformization Theorem for the case of hyperbolic surfaces. A very important result which I will remark, since it emerges as one of its corollaries, is the Riemann Mapping Theorem. An extremely important and widely useful theorem among Complex Analysis which shall become a fundamental object of study for the proceeding sections after the Uniformization Theorem.

Corollary 6 (Riemann Mapping Theorem)**.** *Let $\mathcal{U}$ be a non-empty simply connected open subset of the complex plane $\mathbb{C}$, which is not all of the complex plane.*

Then, there exists a conformal map from $\mathcal{U}$ to the the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Proof. ■

10 The Proof of Uniformization Theorem

Before proceeding with the rest of the proof of the Uniformization Theorem, concluding the cases where the Riemann surfaces are either parabolic or elliptic, let us only establish some minor technical lemmas:

Lemma 10. *Let M be a Riemann surface and D a domain, $D \Subset M$ and a point $p \in D$.*

If f is a meromorphic function on $\overline{D}$ with a single pole of order 1 on p.

Then, there exists a neighborhood N of p such that for any $X_0 \in N$: $f(X) \neq f(X_0)$, $\forall X \in D \backslash \{X_0\}$.

Proof. Since f has a pole of order 1 at p, choosing a local coordinate ϕ near p: $f \circ \phi^{-1}(z) = \frac{1}{z} + g(z)$ (or a change by a multiplying non-zero constant), with g holomorphic. Therefore, computing the derivative:

$$(f \circ \phi^{-1})'(z) = -\frac{1}{z^2} + g'(z).$$

As g' is holomorphic and $-\frac{1}{z^2} \to -\infty$, when $z \to 0$, $(f \circ \phi^{-1})' \neq 0$ near p. So, $f \circ \phi^{-1}$ is injective near p and similarly for f. Denominate this neighborhood of p where f is injective as N_0.

Now, we must prove still that $f(X) \neq f(X_0)$ for $X \in D \backslash N_0$.

For this, let $M_0 = max_{\partial(D \backslash N_0)} |f|$. With the help of the Maximum Principle, we may select a constant $\tilde{N} \in \mathbb{R}$ so big that $\{|z| \geq \tilde{N}\} \cap D \subseteq f(N_0)$. Therefore, choosing $N = f^{-1}(\{|z| \geq \tilde{N}\} \cap D) \subseteq N_0$, $|f(Z)| \geq \tilde{N}$ on N and $|f(Z)| < \tilde{N}$ on $D \backslash \overline{N}$.

Hence, as f is injective on $N \subseteq N_0$ and $|f(Z)| < \tilde{N}$ on $D \backslash \overline{N}$, the lemma is proven.

■

Corollary 7. *Let M be a parabolic or elliptic Riemann surface and a point $p \in M$.*

Suppose f is a meromorphic function on M with a single pole of order 1 on p and such that f is bounded outside one relatively compact neighborhood of p (which by itself and the Maximum Modulus Principle implies it is bounded outside every).

Then, there exists a neighborhood N of p so that for all $X_0 \in N$: $f(X) \neq f(X_0)$, $\forall X \in D \backslash \{X_0\}$.

Proof. ■

Lemma 11. *Consider M a parabolic or compact simply connected Riemann surface and a point $p \in M$.*

Then, there exists a meromorphic function f with a pole of order -1 on p which is bounded outside every neighborhood of p.

Proof. Choose a conformal disc with coordinate z centered about p.

By Theorem 7, there exists a harmonic function u on $M \setminus \{p\}$, bounded outside every neighborhood of p such that $u - \mathfrak{Re}\,\frac{1}{z}$ is harmonic near p and $(u - \mathfrak{Re}\,\frac{1}{z})(p) = 0$. Also, by Corollary 24, there exists a (global) meromorphic function f, holomorphic on $M \backslash \{p\}$, which satisfies: $f = u + iv$ (*i.e.*, $\mathfrak{Re} f = u$), $f - 1/z$ being holomorphic near p and $f - 1/z$ vanishing at p.

Similarly, for the meromorphic function $\frac{i}{z}$, there exists a meromorphic function $\tilde{f}$, holomorphic on $M\backslash\{p\}$, which satisfies: $\tilde{f} = \tilde{u} + i\tilde{v}$, where $\tilde{u}$ is bounded outside every neighborhood of p, $\tilde{f} - i/z$ is holomorphic near p and $\tilde{f} - i/z$ vanishes on p.

Claim. $\tilde{f} = if$. Hence, $v = \tilde{u}$ and v is also bounded outside every neighborhood of p.

Choose a number $m \in \mathbb{R}$ so that: $|u| < m$ and $|\tilde{u}| < m$ on $\{|z| = 1\}$. Of course, by the Maximum Principle for harmonic functions, $|u| < m$ and $|\tilde{u}| < m$ on $\{|z| \geq 1\}$ too (since they are bounded outside this neighborhood). By the previous corollary, we may assume f and $\tilde{f}$ to be injective on $\{|z| \leq 1\}$.

Consider q so that $z(q) \neq 0$ with $|z(q)| < 1$, $|u(q)|, |\tilde{u}(q)| > 2m$ (this is possible since f and $\tilde{f}$ blow up near p) and write the following holomorphic functions on $M\backslash\{q\}$:

$$g(Z) = \frac{1}{f(Z) - f(q)},$$

$$\tilde{g}(Z) = \frac{1}{\tilde{f}(Z) - \tilde{f}(q)}.$$

Let us prove these functions are bounded outside $\{|z| \leq 1\}$ and, as a consequence, outside every neighborhood $\mathcal{U}$ of q (just use the compacity of $\overline{M\backslash\mathcal{U} \cap \{|z| \leq 1\}}$ to see one implies the other).

So, for X with $|z(X)| \geq 1$:

$$|g(X)|^2 = \frac{1}{(u(X) - u(q))^2 + (v(X) - v(q))^2} \leq \frac{1}{m^2}.$$

This is valid since:

$$(u(X) - u(q))^2 + (v(X) - v(q))^2 \geq (u(X) - u(q))^2 \geq$$

$$\geq |\,|u(X)| - |u(q)|\,|^2 \geq (2m - m)^2 = m^2$$

Analogously $|\tilde{g}(X)| \leq \frac{1}{m^2}$, for X with $|z(X)| \geq 1$.

Let us now consider the Laurent expansions of the meromorphic functions g and $\tilde{g}$ in terms of the local coordinate z and around the point $z(q)$:

$$g \circ z^{-1} = \frac{b_{-1}}{z - z(q)} + a_0 + a_1(z - z(q)) + \cdots$$

$$\tilde{g} \circ z^{-1} = \frac{\tilde{b}_{-1}}{z - z(q)} + \tilde{a}_0 + \tilde{a}_1(z - z(q)) + \cdots$$

Hence, the function $b_{-1} \cdot g - \tilde{g} \cdot \tilde{b}_{-1}$ is holomorphic (it only has terms of positive order in its Laurent series) and because of the estimates above, bounded. By Liouville's Theorem and the Identity Theorem for Riemann surfaces, it must be constant all over M. Manipulating this function, we see:

$$\tilde{f}(Z) = \frac{\alpha f(Z) + \beta}{\gamma f(Z) + \eta}, \text{ for certain constants } \alpha, \beta, \gamma, \eta \in \mathbb{C}.$$

Even though the number of constants might seem too large, the knowledge about singularity points and its type will provides us with enough conclusions to verify the claim:

Remember $\left(\tilde{f} - \frac{i}{z}\right)(p) = 0$, hence:

$$\tilde{f} - \frac{i}{z} = \frac{\alpha f(Z) + \beta - \frac{i}{z}\big(\gamma f + \eta\big)}{\gamma f(Z) + \eta}$$

Multiplying by the local coordinate z :

$$\tilde{f}z - i = \frac{\alpha f(Z)z + \beta z - i\big(\gamma f + \eta\big)}{\gamma f(Z) + \eta}$$

Plugging in the point p:

$$0 = \alpha f(p)z(p) - i\big(\gamma f(p) + \eta\big) = \alpha - i\eta - i\gamma f(p)$$

Remember that f also has a singularity at p, hence, this equality implies $\gamma = 0$. Besides, from the same relation, we must have $\alpha = i\eta$, too.

Thus, obtaining:

$$\tilde{f}(Z) = \frac{i \cdot \eta f(Z) + \beta}{\eta}, \text{ for certain constants } \beta, \eta \in \mathbb{C}.$$

Of course, we can divide the nominator and denominator by η:

$$\tilde{f}(Z) = i \cdot f(Z) + \beta, \text{ for certain constants } \beta \in \mathbb{C}.$$

Once again, subtracting by $\frac{i}{z}$ and then applying the expression on p results in:

$$\tilde{f} - \frac{i}{z} = i \cdot f(Z) + \beta - \frac{i}{z}$$

$$0 = i \cdot 0 + \beta \Longleftrightarrow \beta = 0$$

In conclusion, $\tilde{f} = if$, so $v = \tilde{u}$ and v as well as f is bounded outside every neighborhood of p.

■

10.1 Admissible functions

From the above theorem, we have established the existence of an important class of functions which, henceforth, shall be denominated as the class of admissible functions (at p):

Definition 36 (Admissible functions)**.** *Consider $p \in M$, where M is a Riemann surface.*

A function g shall be called admissible at the point p if the following are verified:

(i) g is holomorphic on $M\backslash\{p\}$.

(ii) the singularity at p is a pole of order -1.

(iii) g is bounded outside every neighborhood of the point p.

These functions serve as very appealing transformations, not only for their good behavior outside the singularity but also because of their special properties, in particular, the manner they relate to each other and the injectiveness inherent to them, as we will see below. These properties will be the last steps before finishing our proof of the Uniformization Theorem.

Theorem. *Suppose f is an admissible function at the point p and g is another admissible function at a different point q.*

Then, there exists a Möbius transformation $\mathcal{L}$ so that $f = \mathcal{L} \circ g$.

Proof. Consider the function f described as above and the set:

$$\tilde{M} = \{\, X \in M \mid\ g \text{ is admissible at } X \Rightarrow \exists \text{ Möbius function } \mathcal{L} : g = \mathcal{L} \circ f \,\}$$

$\tilde{M}$ is non-empty since $p \in \tilde{M}$, following Lemma 11. Let us show $\tilde{M}$ is open and closed.

Choose any $q \in \tilde{M}$ and an admissible function $\hat{g}$ on q. Thus, there exists a Möbius transformation L such that $\hat{g} = L \circ f$. Remembering the previous demonstration, consider a small neighborhood N of q (which we shall prove belongs to $\tilde{M}$) and any point $q_1 \in N$. Then, define the function:

$$g_1(Z) = \frac{1}{\hat{g}(Z) - \hat{g}(q_1)}\,.$$

g_1 becomes an admissible function at q_1, which is the composition of a Möbius transformation: L_1, with $\hat{g}$.

Now, consider h any admissible function at q_1. Following exactly the proof of Lemma 11, there exist constants $c, \tilde{c} \in \mathbb{C}$ such that $c\, g_1 - \tilde{c}\, h$ is holomorphic and bounded on an open set, hence, constant on M.

Thus, if h is admissible at q_1, then there is another Möbius transformation L_2 so that $h = L_2 \circ g_1 = L_2 \circ L_1 \circ \hat{g} = (L_2 \circ L_1 \circ L) \circ f$. As the composition of Möbius transformations is again a Möbius transformation, $\tilde{M}$ is, in fact, open.

Analogously, using the characterization of closed sets in metric spaces through limit points of sequences, easily, we also know that $\tilde{M}$ is a closed set since if the point q is the limit of a sequence of points $(q_n)_n$ in $\tilde{M}$, then choose a small neighborhood of q and apply the same exact reasoning as above.

∎

Theorem. *Every admissible function is injective.*

Proof. Assume f is an admissible function at a point p. We say $f(p) = \infty$.

Let us imagine that $f(a) = f(b)$. Choose an admissible function h at a. By the last theorem, there exists a Möbius transformation L such that $f = L \circ h$.

Thus, $(L \circ h)(a) = (L \circ h)(b)$. As Möbius transformations are always injective, $h(b) = h(a)$ which equals ∞ (since h is an admissible function at a).

However, as h is an admissible function at a, there exists only a single point at which h achives the value ∞, therefore, $a = b$.

■

We will now, as promised, conclude the proof of the Uniformization Theorem:

Theorem 25 (The End of Uniformization Theorem)**.** *Let M be a simply connected Riemann surface.*

If M is compact, then M is conformally equivalent to the Riemann sphere $\mathbb{C}_\infty$.

Besides, if M is parabolic, M is conformally equivalent to the complex plane $\mathbb{C}$.

Proof of the Riemann Uniformization Theorem:

Suppose, firstly, that M is compact.

Consider an admissible function on M, say $f : M \to \mathbb{C}_\infty$. As f is a meromorphic function on M, f will be a holomorphic function onto $\mathbb{C}_\infty$. By the last theorem, we also know f is injective.

Now, since f is non-constant and holomorphic, f is an open map. Hence, $f(M)$ is open and because M is compact, $f(M)$ is also closed. We conclude, by connectedness, that $f(M) = \mathbb{C}_\infty$. Thus, f is a holomorphic and bijective function onto $\mathbb{C}_\infty$, *i.e.*, a conformal map.

Suppose, now, that M is parabolic.

If f was surjective, then f would be a conformal map from M to $\mathbb{C}_\infty$. This is a contradiction since one is compact and the other is not.

Consequently, f is misses at least one point of $\mathbb{C}_\infty$. Remembering that Möbius transformations are always conformal, we can assume $L \circ f(M) \subseteq \mathbb{C}$ for some Möbius transformation L.

Claim. Suppose D is a simply connected domain on $\mathbb{C}_\infty$.

If $\mathbb{C}_\infty \backslash D$ contains 2 or more points, that is, D misses 2 or more points of $\mathbb{C}_\infty$, then D is a hyperbolic Riemann surface.

Proof of Claim. We may assume D does not contain the points 0 and ∞ (by composing with a Möbius transformation).

Therefore, the (non-vanishing) function $Id_{|D}$ has a holomorphic square root on D (as D is assumed to be simply connected), *i.e.*, a holomorphic function g on D such that $g^2 = Id$.

Let $w \in D$ and $\sqrt{w} = g(w)$.

We will now verify that g does not attain the value $-\sqrt{w}$. If it did, then there is $z \in D$ for which $g(z) = -\sqrt{w}$ and so $z = g^2(z) = (\sqrt{w})^2 = g^2(w) = w$. This is a contradiction since it implies $g(w) = -\sqrt{w} = \sqrt{w}$ even though we assumed $w \neq 0$ and also shows g misses a ball centered at $-\sqrt{w}$, say, of radius ϵ.

So, we may define function:

$$h(z) = \frac{1}{g(z) + \sqrt{w}} \ , \quad z \in D.$$

h is holomorphic and since g misses the ball of radius ϵ centered on $-\sqrt{w}$, then $|h| \leq \frac{1}{\epsilon}$, so it is bounded. Hence, taking its real part and adding $1/\epsilon$, we would have a harmonic, non-constant and positive function. That is, D must be hyperbolic. □

Remark. This is an analogous for Riemann surfaces to Picard's Little Theorem on the complex plane.

Now, as $L \circ f$ is a conformal map onto its image $L \circ f(M)$, it must be a simply connected and a parabolic surface. This means, by the claim, we can't have $L \circ f(M) \subsetneq \mathbb{C}$. Hence, $L \circ f(M) = \mathbb{C}$ and we have a conformal map from M, a parabolic surface, to $\mathbb{C}$. This, finally, ends our proof.

■

11 Circle Packings and Convergence to the Riemann Mapping

After establishing a result as powerful as the Riemann Uniformization Theorem, we see that (at least on the simply connected case) there is actually only a very selected amount of Riemann surfaces one has to work with. Most of the times, thanks to this theorem, we may, whenever necessary, transport a question on generic Riemann surfaces inside the plane to the 2-sphere, the disc or $\mathbb{C}$ itself. However, when dealing with this theory, there exist occasions under which we may be interested in the explicit expression of the conformal mapping given by the Uniformization Theorem (or an approximation).

In these cases, it's clear that the raw proof by itself won't bring much of a help due to its theoretical nature. Hence, the purpose of the remaining of this dissertation shall be to mix the areas of Complex Analysis and the Theory of Graphs in order to obtain a beautiful method of approximating these conformal maps. To be more precise, our new purpose will be to estimate the function present in the Riemann Mapping Theorem 6.

Let us now introduce/remind the reader of some basic concepts in the Theory of Graphs which, in surprising and unexpected ways, will be used throughout this and the upcoming sections:

Definition 37. *A graph is an ordered pair $G = (V, E)$ consisting of a set of vertices $V = (v_n)_{n\in\mathbb{N}}$ and a set of edges E in $\{\{v_k, v_l\} : v_k, v_l \in V$ and $v_k \neq v_l\}$.*

If an edge e is of the type $\{\cdot, v\}$, for some vertex v, then we say *e is an incident edge to v.* Exploring further, many different categories of graphs arise:

Definition 38. *A simple graph is one that doesn't have multiple edges or loops.*

We will always assume all graphs in this dissertation are simple and planar too, that is, one may represent these in the plane through a graph with no crossing edges.

Definition 39. *A simple graph G is connected if, for every two vertices $v, w \in G$, there exists a finite sequence of vertices $(v_1, \cdots, v_m)$ such that $v_1 = v$, $v_m = w$ and $\{v_i, v_{i+1}\} \in E$, $\forall i \in \{1, \cdots, m-1\}$.*

In other words, a graph G is connected if, for every two vertices $v, w \in G$, there is a path from the initial vertex v to w.

Definition 40. *Consider a simple planar graph G (embedded on the complex plane).*

A face on G is one connected component of $\mathbb{C} \backslash G$.

Notice this definition includes always exactly one "unbounded face" by the Jordan Curve Theorem.

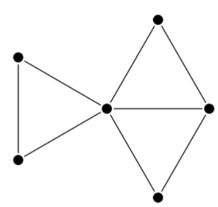


Figure 9: As an example, the graph above has four faces, three bounded and formed by triangles and one unbounded, adjacent to seven edges.

Finally, in order to conclude our small digression through the world of graphs, we define a very special category: the plane triangulations.

Definition 41. *A graph is called a plane triangulation if all of its faces are triangles, that is, bounded by three edges (notice this definition includes the outer face!).*

Looking at Figure 9, one may assume at first sight that the graph in question is, in fact, a plane triangulation. However, this is not the case since the outer face is actually one heptagon.

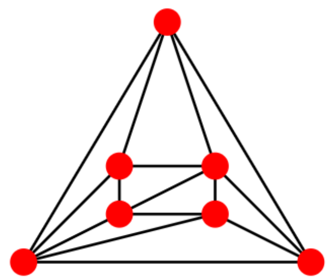


Figure 10: An example of an actual plane triangulation (where even the outer face is bounded by three edges).

11.1 Circle Packings

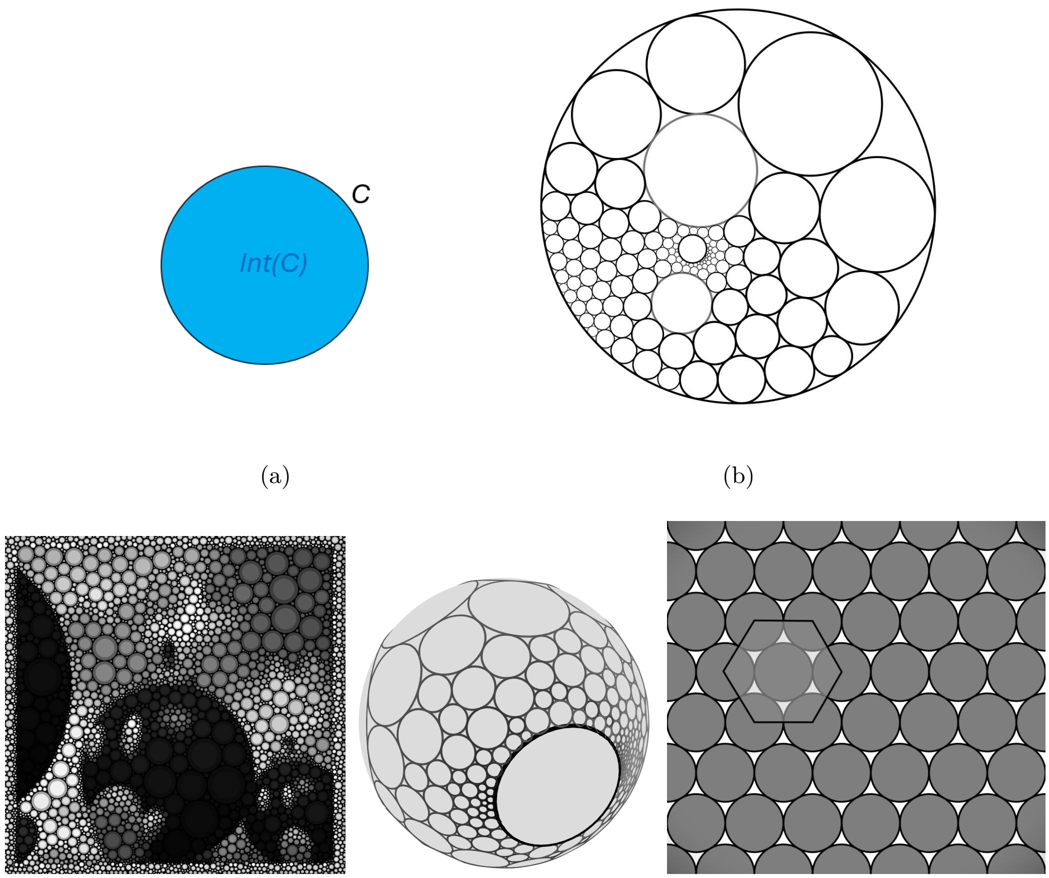
Suppose our ambient space is the complex plane $\mathbb{C}$.

A *circle centered at* z_0 *with radius* r_0 is the set: $\{z \in \mathbb{C} : |z - z_0| = r_0\}$.

Suppose C is a circle. Then, one defines the *interior of the circle* C, $Int(C)$, as the bounded connected component of $\mathbb{C} \backslash C$. Besides, $D(C)$, *the disc associated to* C, will represent the union between C and its interior: $D(C) = C \cup Int(C)$.

Definition 42. *A circle packing is a collection* $P = (P_j)_{j \in \Lambda}$, *with* Λ *a finite or countable index, of circles which possess mutually disjoint interiors (i.e., can only intersect tangentially) and whose union forms a connected set.*

Many examples can be illustrated:



(a) (b)

(c) (d) (e)

Figure 11: An example of a circle, C, an its interior, Int(C), presented in (a), along with four examples of circle packings. In (d), we see a circle packing transported to the unit sphere through stereographic projection and in the last one, (e), the densest circle packing on the plane: the regular hexagonal packing.

Now, before proceeding with the theory and important results regarding circle packings, one must first be familiar with the vocabulary adjacent to this subject:

Definition 43. *Consider P a circle packing and $c \in P = (P_j)_{j\in\Lambda}$, a circle.*

An interstice is one connected component of the complement of $\bigcup_{j\in\Lambda} D(P_j)$ $(= P_j \cup Int(P_j))$.

The flower centered at c is the union of $D(c)$ (i.e., c together with its interior), all circles tangent to c and their interiors together with the interstices formed by all of these circles.

The support of the circle packing P, supp(P), is the union of all discs at p and their bounded interstices.

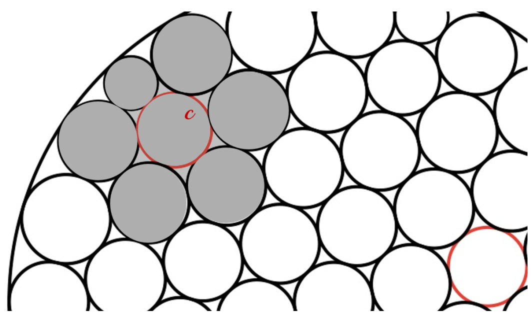


Figure 12: A circle, c , and its flower (in gray).

With every circle packing, comes a new important concept for our research, its associated *nerve*:

Definition 44. *Consider a circle packing $P = (P_j)_{j\in\Lambda}$.*

Its associated contact graph (also called tangency graph or nerve) is the graph $G = (V, E)$ whose vertex set, V, equals the centers of the circles P_j and whose edges, E, are such that $\{v_k, v_l\} \in E$ if and only if the circles P_k and P_l have non-empty intersection.

The interior of the nerve is defined as the union between the nerve itself and its bounded faces (thinking of the nerve as a graph). This concept is also known as the carrier of a circle packing P, denoted $Carr(P)$.

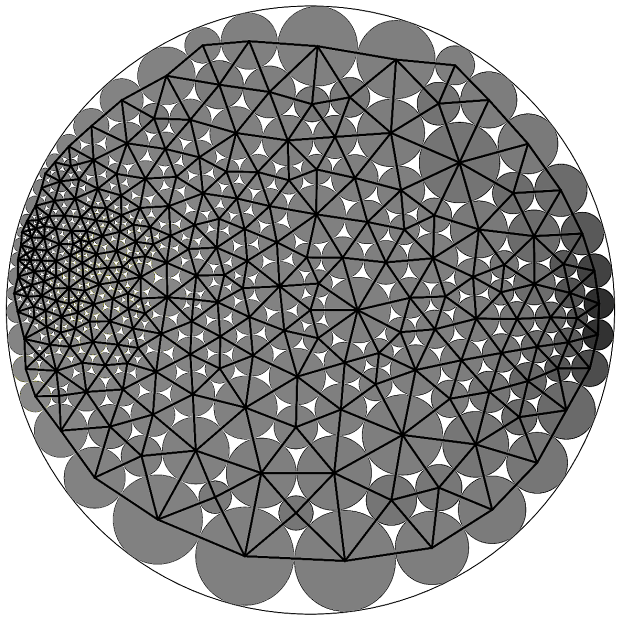


Figure 13: Example of a circle packing and its nerve/contact graph/tangency graph (represented through the lines in black).

Notice that the simply connectedness of the support of a circle packing is equivalent to that of its carrier.

Of course, it is very simple to construct and draw the tangency graph of some circle packing.

However, naturally, another interesting question arises: given a finite (planar, simple) graph G, is there a circle packing whose nerve equals G? The research topic of circle packings and, in particular, the search for answers to this problem garnered the special attention of three important mathematicians: Paul Koebe, E. M. Andreev and William Thurston.

The theorem that follows, known as the Circle Packing Theorem, was proved for the first time by Koebe and later rediscovered by William Thurston following Andreev's studies and research. Thanks to their contributions we have an affirmative answer to the preceding question:

Theorem (Circle Packing Theorem)**.** *Suppose G is a finite, planar, simple graph. Then, there exists a circle packing on $\mathbb{C}$ whose nerve is precisely the graph G. Furthermore, if G is a plane triangulation, the circle packing is unique up to composition by reflections and Möbius transformations.*

The proof of this theorem can originally be found at [6], although a simpler demonstration is presented in the thesis [7].

However, this is not the central point of interest in our investigation. In fact, circle packings may actually be manipulated in order to analyze the behavior of bi-holomorphic mappings onto the unit disc D, as in the content of the Riemann Mapping Theorem 6. The details will be shown from now on:

Suppose R is an open, simply connected and bounded region on $\mathbb{C}$.

Consider a finite circle packing P covering the region R in question with all bounded interstices being triangular. A circle $c \in P$ is called an *inner circle* if its flower lies inside R and a *boundary circle* if c is not inner but it is tangent to an inner circle. Lastly, the *border* of the circle packing is the union of all boundary circles.

Now, in order to create a notion of convergence using this new theory, consider a particular sequence of circle packings, $(P^n)_n$, given by the regular hexagonal packing about R:

$$P^n = \frac{1}{n} \cdot \left[\bigcup_{x,y \in \mathbb{Z}} (2x + 2e^{2\pi i/3} y) + S^1 \right]$$

Now, select a circle packing C in the sequence P^n and consider its nerve: T. Creating a distinguished point ∞ and drawing aditional lines from ∞ to the centers of the border circles, we may complete T and obtain a topological triangulation, T^*, of the sphere S^2 (using stereographic projection to travel from $\mathbb{C} \cup \{\infty\}$ to S^2).

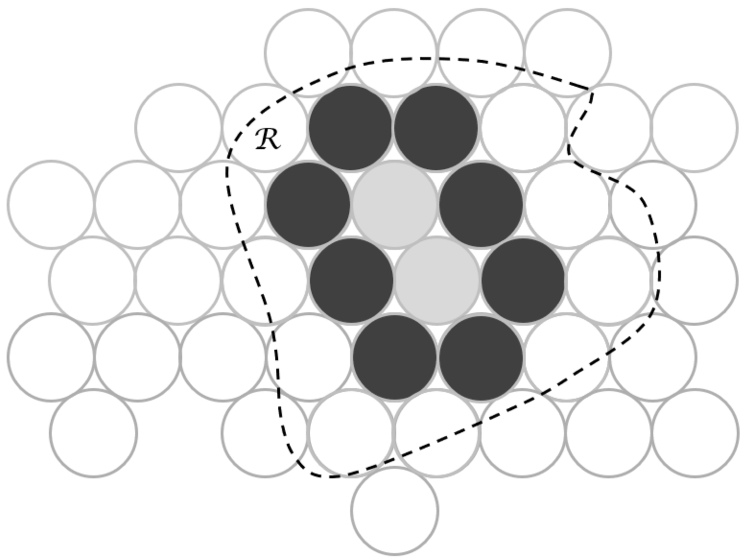


Figure 14: A region $R \subseteq \mathbb{C}$ bounded by the dashed lines, together with a circle packing around it and its (two) inner and (eight) boundary circles.

Invoking the Circle Packing Theorem, one may create a circle packing C^* whose nerve equals T^*, unique up to composition by Möbius transformations.

Remembering the existence of Möbius transformations that send the interior of a specified circle to the exterior of another (inversions), proceed by composing the preceding circle packing with a Möbius transformation that sends the interior of the circle centered at ∞ to the exterior of the unit disc D and return the result to the complex plane.

We may further normalize this circle packing, using a Möbius transformation which fixes the unit disk in a way that sends a specific point $z_1^* \in R$ to the origin and another point, $z_2^* \in R$, becomes centered on the positive real axis (later the reasons for this normalization process will become clearer, although this procedure should remind the reader of the uniqueness conditions for the Riemann Mapping Theorem).

Call this final circle packing C'. Resuming what happens in the end, we have transported a circle packing from the initial region R (on $\mathbb{C}$) to a circle packing on the open unit disc D, where all border discs in R become internally tangent to ∂D (that is, tangent to the unit circle). Furthermore, these circle packings have *isomorphic* carriers, in the sense that there exists a homeomorphism $\phi : Carr(C) \to Carr(C')$.

This homeomorphism corresponds to the *simplicial map* induced by the association between centers of circles in C and C^*. In order to comprehend the construction of the latter, firstly, write $V(C)$ as the set of centers of circles on C and $V(C')$ the set of centers on C'. We have a bijective mapping $f : V(C) \longrightarrow V(C')$.

The $affine\ extension/simplicial\ map$ of f is the mapping $|f| : |V(C)| = Carr(T) \longrightarrow |V(C')| = Carr(T')$ defined as follows:

Any point $x \in supp(C)$ has a unique representation $x = \sum_{v_i \in V(C)} a_i\, v_i$ with with $a_i \geq 0$ and $\sum_{i=0}^{k} a_i = 1$. Thus, $|f|$ can be constructed as $|f|(x) := \sum_{v_i \in V(C)} a_i\, f(v_i)$.

Notice that thanks to the bijectivity of the correspondence between vertices, the affine extension $|f|$ is guaranteed to be, in fact, a homeomorphism. This homeomorphism clearly also establishes a (bijective) correspondence between circles in C and circles in C'.

Hence, iterating this process for the whole sequence $(P^n)_n$ shall provide us with a sequence of homeomorphisms from circle packings in R to the corresponding ones inside the unit disc D. We will name this homeomorphisms as the *approximating Riemann maps* and denote them by $\phi_n : Carr(P^n) \to Carr(P'^n)$. These also satisfy $\phi_n(P^n) = P'^n$, as mentioned before.

After establishing this construction, W. Thurston originally conjectured that the sequence of homeomorphisms considered above converges uniformly on compacts to a conformal homeomorphism from R to D, *i.e.*, a Riemann mapping. The objective of the research from this point onwards will be to demonstrate this result actually holds.

Notice that, through the perspective of applications, this can be very useful since if one wishes to visualize a conformal map from any simply connected, bounded and open set on the plane it suffices to implement an algorithm with the purpose of creating circle packings covering the region and, after this, observe their respective images (instead of, for example, using the whole proof of the Uniformization Theorem just to find it).

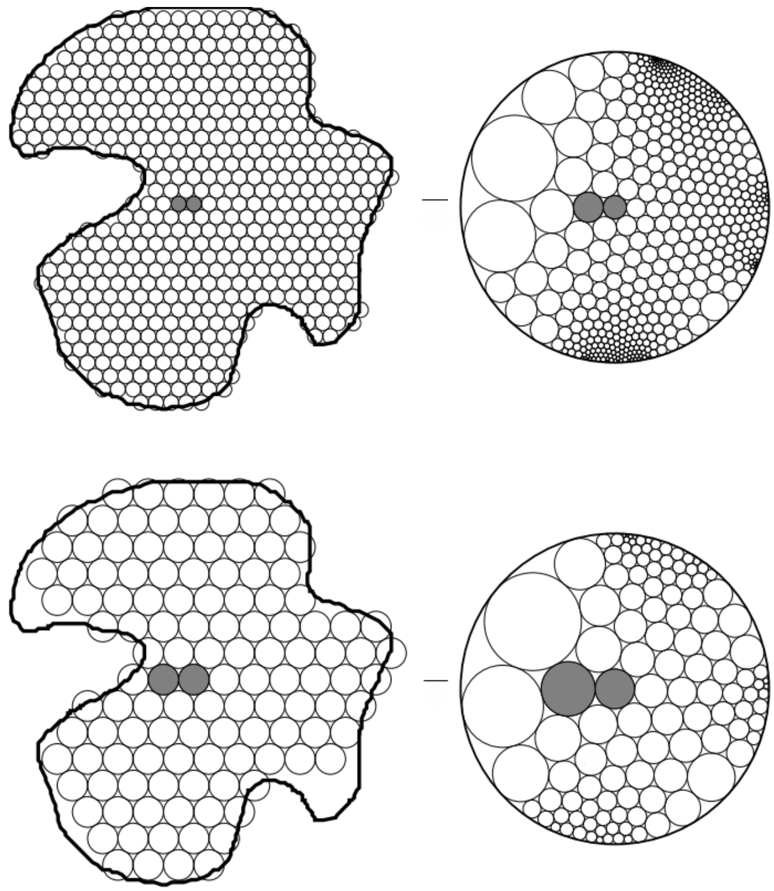


Figure 15: An approximation to the Riemann mapping of a region through hexagonal circle packings.

12 An Introduction to Quasi-conformal mappings

12.1 A new perspective on Conformal Maps

Before advancing, I would like to remark that the term "conformal mapping" has been used throughout as defined in the beginning of this paper: a bijective and holomorphic function. However, as good and practical as this analytic definition is, when seeing these maps through the perspective of circle packings, another equivalent geometric formulation becomes even more useful:

Definition 45. *Suppose $\Omega \subseteq \mathbb{C}$ is an open domain.*

A map $f : \Omega \longrightarrow \Omega' \subseteq \mathbb{C}$ is called angle and orientation-preserving if and only if, it is differentiable, for every point $z = x + iy \equiv (x, y) \in \Omega$ the derivative $Df_{(x,y)}$ has positive determinant (i.e., preserves orientation) and the derivative matrix $Df_{(x,y)}$ preserves angles, meaning (by results in linear algebra) that there exists a real constant $p = p(x, y) > 0$ such that ${Df_{(x,y)}}^T \cdot Df_{(x,y)} = p(x, y) \cdot Id$.

Remark. In some nomenclatures, for example, in Wikipedia, this is the usual definition of the term "$conformal$" which does not coincide with the one used in this dissertation paper.

Now, an amazing connection between the preceding notion and the bi-holomorphic maps emerges in the form of the following theorem:

Theorem. *Suppose $\Omega \subseteq \mathbb{C}$ is an open set and let $f : \Omega \longrightarrow \Omega' \subseteq \mathbb{C}$ be a real-differentiable map (seen as a map between open sets of $\mathbb{R}^2$).*

Then, f is angle and orientation-preserving if and only if f is locally invertible and complex analytic, that is, a holomorphic map whose derivative never vanishes.

Proof. Firstly, suppose f is holomorphic and its derivative never vanishes. Let us prove the derivative of f at every point is angle-preserving using curves to represent the tangent vectors.

So, consider two curves γ_1 and γ_2 such that $\gamma_1(0) = \gamma_2(0) = z_0 \in \Omega$. Our purpose is to show the angle between the tangent vectors $\gamma_1'(0),\ \gamma_2'(0)$ is the same as the angle between $(f \circ \gamma_1)'(0)$ and $(f \circ \gamma_2)'(0)$.

As f is holomorphic and never vanishes, then:

$$\measuredangle\Big\{(f \circ \gamma_1)'(0), (f \circ \gamma_2)'(0)\Big\} = arg\left(\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)}\right) = arg\left(\frac{f'(\gamma_1(0)) \cdot \gamma_1'(0)}{f'(\gamma_2(0)) \cdot \gamma_2'(0)}\right)$$

As the derivative never vanishes:

$$\measuredangle\Big\{(f \circ \gamma_1)'(0), (f \circ \gamma_2)'(0)\Big\} = arg\left(\frac{\gamma_1'(0)}{\gamma_2'(0)}\right) = \measuredangle\{\gamma_1'(0), \gamma_2'(0)\}$$

Therefore, Df_{z_0} is angle-preserving, for all $z_0 \in \Omega$.

Thanks to the Cauchy-Riemann Equations, one can also easily verify Df_{z_0} also has positive determinant equal to $\left|f'(z_0)\right|^2 > 0$. Thus, the map f is conformal.

For the reciprocal, let f be a differentiable function between open sets of $\mathbb{C} = \mathbb{R}^2$ that satisfies, for any $z = (z_1, z_2) \in \Omega$: ${Df_{(z_1,z_2)}}^T \cdot Df_{(z_1,z_2)} = p(z_1, z_2) \cdot Id$ with its determinant always positive.

Writing $Df_{(z_1,z_2)}$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ba+dc & b^2+d^2 \end{pmatrix} = p \cdot Id$$

In particular, $a^2 + c^2 = p$ and $b^2 + d^2 = p$, so there exist $\theta, \varphi \in [0, 2\pi]$ such that $a = \sqrt{p} \cdot cos(\theta)$, $c = \sqrt{p} \cdot sin(\theta)$, $b = \sqrt{p} \cdot cos(\varphi)$ and $d = \sqrt{p} \cdot sin(\varphi)$. The second and third equations coming from the matrix imply that:

$$p \cdot cos(\theta)cos(\varphi) + p \cdot sin(\theta)sin(\varphi) = 0 \Longleftrightarrow cos(\theta - \varphi) = 0$$

So, $\theta = \varphi \pm \frac{\pi}{2}$ and Df_z can only have the two aspects:

$$Df_z = \sqrt{p} \begin{pmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta) \end{pmatrix} \text{ or } Df_z = \sqrt{p} \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

The first case corresponding to a scaled reflection and the second a scaled rotation. Furthermore, as f preserves orientation, the only possible hypothesis is that Df_z must always be a scaled rotation for every point z.

Moreover, using the notation for f as real function $f(z) \equiv f(x,y) = \mathfrak{Re}(f)(x,y) + i \cdot \mathfrak{Im}(f)(x,y) \equiv u(x,y) + iv(x,y)$, one must have that:

$$Df_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

Thus, we conclude $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, the Cauchy-Riemann equations. As a consequence, f is holomorphic everywhere and because the determinant of Df_z is $\left|f'(z)\right|^2$ and p, with p never vanishing, it also implies f' is always non-zero.

■

Hence, we see that bi-holomorphic maps correspond exactly to bijective angle and orientation-preserving functions and we may interchange between the complex and the geometric perspectives of conformality. Therefore, from this point forwards, I will apply both of these views as being one and the same.

Notice that, in terms of the circle packing subject, by Taylor's Theorem, we know $f(z) \sim f(z_0) + f'(z_0) \cdot (z - z_0)$ near z_0, so the behavior of f will be similar to that of $f'(z_0) = Df_{z_0}$. In the conformal case, this translates to f sending small circles to "approximate circles". Consequently, the condition of bijective conformal maps (from the geometric point of view) will make these maps send very small circle packings to other "approximate circle packings".

12.2 Quasi-Conformal Mappings

In order to advance on this matter, we shall first relax the strict notion of conformality and "shape-preserving" maps to functions that just "dilate" these shapes, the so called $K - quasiconformal$ *mappings*.

Definition 46. *Let* $K \in [1, \infty)$. *Consider a map* $f : \Omega \to \Omega'$ *between two domains* Ω, Ω' *in the complex plane.*

For any point $z_0 \in \Omega$, *we define:*

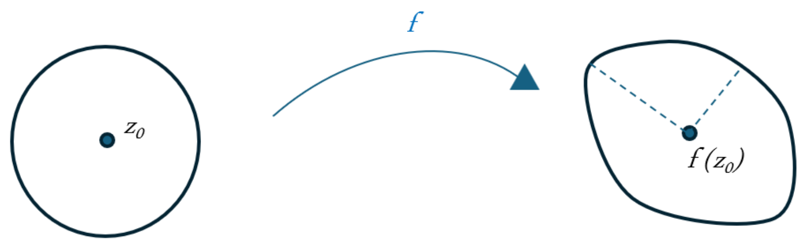
$$L_f(z_0, \delta) = max\{\, |f(z) - f(z_0)| : z \in \partial B(z_0, \delta)\,\}$$

$$l_f(z_0, \delta) = min\{\, |f(z) - f(z_0)| : z \in \partial B(z_0, \delta)\,\}$$

The mapping $f : \Omega \to \Omega'$ *is* $K - quasiconformal$ *if and only if for all* $z_0 \in \Omega$:

$$d_f(z_0) = limsup_{\delta \to 0} \left(\frac{L_f(z_0, \delta)}{l_f(z_0, \delta)} \right) \leq K.$$

The term $d_f(z_0)$ is called the *dilation/dilatation of f at z_0*. It is also sometimes denoted as $H_f(z_0) = H(z_0)$.



Remark 9. A map is called quasi-conformal if it is $K - quasiconformal$, for some $K \in [1, \infty)$.

Remark 10. Standard facts about quasi-conformal maps include the knowledge that composition maintains quasi-conformality and that the inverse of a bijective quasi-conformal function is also quasi-conformality.

Perhaps of even bigger importance is the local property underlying quasi-conformality:

Theorem 26 (Local Property of Quasi-Conformal Mappings)**.** *Consider $K \geq 1$ and an orientation-preserving homeomorphism $f : U \to V$ between complex domains. If, for every $z_0 \in U$ there exists a neighbourhood U_{z_0} of z_0 in U such that $f|_{U_{z_0}} : U_{z_0} \to f(U_{z_0})$ is K-quasiconformal, then f itself is K-quasiconformal.*

Proof. As the definition of quasi-conformality on f only depends on H_f being bounded for every point and this measure is clearly local, the theorem follows.

■

Intuitively, 1-quasiconformality corresponds to non-existence of angle distortion, so one could, naturally, think that 1-quasiconformal maps will be conformal and, in fact, an important result establishes that a homeomorphism (between open subsets in the complex plane) is 1-quasiconformal if and only if it is conformal or anticonformal (preserves angles but reverses orientation), a theorem due to Menshov from 1937.

Besides, an important basic result about quasi-conformality maps is the fact these are well-behaved relative to collages in the sense presented by the theorem:

Theorem 27. *Let $K \geq 1$.*

Suppose $f : U \to V$ is an orientation-preserving homeomorphism and Γ is a finite union of real analytic lines that form a closed curve lying inside U, except possibly at the beginning and end point. If the restriction $f\big|_{U \setminus \Gamma}$ is K-quasiconformal, then f is also K-quasiconformal.

Another important remark is that in the definition of quasi-conformal mappings, we have made use of the circle shape but this can be equivalently defined through annulus or even triangles. We'll inspect more deeply this last form of evaluating quasi-conformality. In the book of John Hubbard, [4], a new characterization of quasi-conformal mappings using triangles was given as below.

Definition 47. *Let T be a topological triangle on the complex plane, with vertices v_1, v_2, v_3. We define:*

$$L(T) = max\{\ |a-b| : a, b \ are\ vertices\ of\ T\ \ \}$$

$$\ell(T) = min\{\ |a-b| : a, b \ are\ vertices\ of\ T\ \ \}$$

Then, the skew of the triangle T is the quotient between these two quantities:

$$skew(T) = \frac{L(T)}{\ell(T)}$$

J. Hubbard concluded that a homeomorphism between two domains in the complex plane $f : U \to V$ is quasi-conformal provided there exists an increasing homeomorphism $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$skew(f(T)) \leq h(skew(T)), \text{ for all closed Euclidean triangles } T \subset U.$$

In fact, Hubbard even had the intention of generalizing this result and prove it suffices to restrict ourselves to equilateral triangles. This question was later answered affirmatively by Colleen Ackermann, Peter Haïssinsky and Aimo Hinkkanen, together with a new discovery, Theorem 2 in their paper [1], which provides a very useful way to verify quasi-conformality of a mapping. Citing:

Theorem 28. *Suppose R is a domain in the complex plane and that $f : R \to \Omega$ is an orientation-preserving homeomorphism.*

Define $Skew(f)$ as the supremum of $skew(f(T))$ taking all the equilateral triangles $T \subseteq \Omega$.

Besides, let $z \in R$ and $r > 0$. $skew(f, z, r)$ will denote the supremum of $skew(f(T))$ over all equilateral triangles $T \subset B(z, r)$. Set $skew_f(z) = liminf_{r\to 0}\ skew(f, z, r)$ and $skew(f) = ||skew_f||_\infty$.

Suppose that each $z \in U$ has a neighborhood W such that $Skew(f|_W)$ is finite and that there exists a constant $\sigma \geq 1$ satisfying: $skew(f) \leq \sigma$.

Then, we may conclude that f is $K(\sigma){-}quasiconformal$, for a constant $K = K(\sigma) \geq 1$ given by:

$$K(\sigma) = \frac{\sigma^2 - 1 + \sqrt{\sigma^4 + \sigma^2 + 1}}{\sqrt{3}\sigma}.$$

Thus, in the particular case skew$(f) = 1$*, the constant* $K(\sigma)$ *equals* 1 *and the map is conformal.*

Therefore, for the simpler case where the *skew* of all equilateral triangles is less or equal to $\sigma \geq 1$, we may conclude immediately that f is quasi-conformal. Furthermore, notice that, as quasi-conformality on a point is a local property, we can also only consider "very small" (infinitesimal) triangles to verify this characteristic (one should keep in mind that even small triangles can have large skew).

This will become an essential step towards proving that the discrete maps from circle packings converge to a conformal mapping.

I would like to add that the theory behind quasi-conformal mappings is vast and expanding in its research, there exist many more ways to define this concept and properties not covered in this dissertation. Special emphasis on the different equivalent definitions that involve the (almost-everywhere) C^1 properties underlying in this homeomorphisms. However, I'll only focus on the necessary aspects described above, which shall be the only ones used throughout.

13 Some Geometric Lemmas

We will now start the proof of the convergence of the approximating maps to a conformal mapping onto the open unit disc D, beginning with three interesting geometric lemmas:

Lemma 12 (The Ring Lemma)**.** *An ordered finite sequence of N circles $\{c_1, \cdots, c_N\}$ in a circle packing is called a cycle if, for $i \in \{1, ..., N-1\}$: c_i is tangent to its successor, c_{i+1}, and the last circle is also tangent to the first (with tangency points always different).*

For $n \geq 3$, if n circles form a cycle externally tangent to the unit disc (i.e., the circles surround the unit disc), then, there exists a constant $r = r(n)$, depending only on n, such that each circle has radius at least r_n.

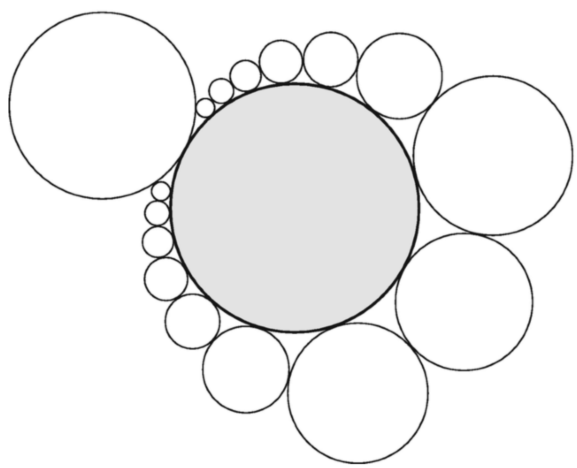


Figure 16: A cycle of $n = 16$ circles surrounding the unit disc.

Proof. Suppose n circles, $\{c_1, \cdots, c_n\}$, surround the unit disc as described. Say $\tilde{c}_1$ is the largest of these, that is, the one with largest radius.

Consider the case where n circles of the exact same radii surround the unit disc, let's call this the regular configuration of n circles. What happens if we try to increase the size of one of them, is that necessarily another will decrease its radius, as Figure 17 illustrates. Hence, a lower bound for the largest circle's radius is the length of the radii when the surrounding configuration is regular.

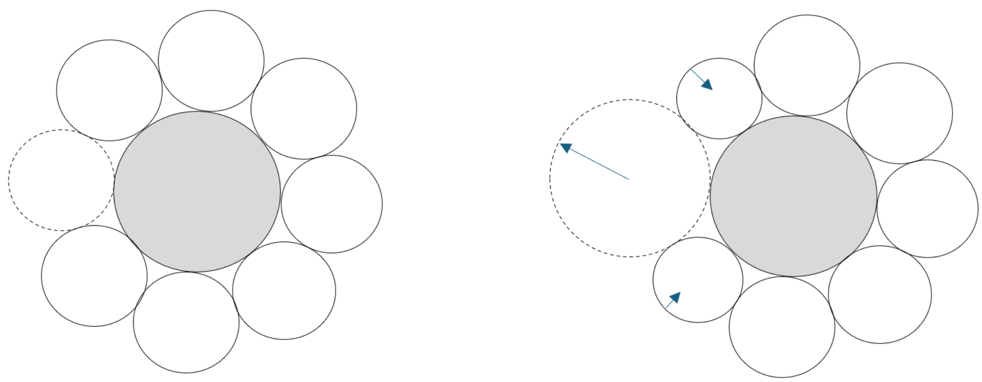


Figure 17: Ring lemma illustration.

Now consider the case where $n = 3$ with the radii of the surrounding circles respectively equal to R_1, R_2 and R_3. In this case, Descartes' Theorem states that:

$$\left(1 + \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^2 = 2 \cdot \left(1 + \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2}\right)$$

Hence, we see that none of the radii R_i, $i \in \{1, 2, 3\}$ can become arbitrarily close to zero.

For the case $n > 3$ where not all radii are equal, proceed as follows. Suppose once again that $\tilde{c}_1$ is the biggest circle that surrounds c_0. Call $\tilde{c}_2$, $\tilde{c}_3$, etc, the adjacent circles from right to left. Let us make the radius of $\tilde{c}_2$ approach zero.

What will happen is that the circles $\tilde{c}_2$ and $\tilde{c}_3$ will slide inside the bounded space between $\tilde{c}_1$ and c_0, as shown by the proceeding picture.

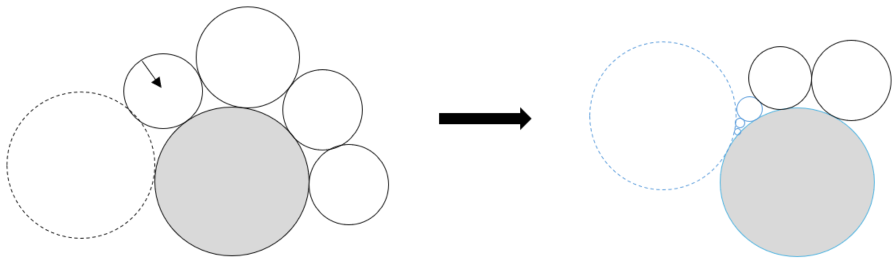


Figure 18: Decreasing the radii of circles surrounding c_0 (notice the circles on the left around the smallest circle).

Hence, recalling the case $n = 3$, we know the radius of $\tilde{c}_2$ must have a lower bound. Repeating this process for the remaining circles, we end up with the constants r_n from the Ring Lemma statement.

■

A deeper dive on this subject, more particularly, on the sharp ring lemma is investigated on the thesis [9]. Interestingly, in this paper, it is shown that the optimal constants for the r_n, with $n \in \mathbb{N}$, are exactly equal to $(F_n^2 + F_{n+1}^2 - 1)^{-1}$, where F_n is the n-th term on the Fibonacci sequence $(1, 1, 2, 3, 5, 8, \cdots)$. Hence, for example, when $n = 6$ circles surround the unit disc, their radii are bounded below by $\frac{1}{232}$.

Lemma 13 (The Length-Area Lemma)**.** *An ordered finite sequence of N circles $\{c_1, \cdots, c_N\}$ in a circle packing is called a chain if, for $i \in \{1, ..., N-1\}$: c_i is tangent to its successor c_{i+1} (note that the last needs not to be tangent to the first). The (combinatorial) length of a chain is the number of circles that composes it.*

Let P be a packing of D whose border circles are internally tangent to ∂D and let $c \in P$ be a circle.

Suppose T_1, T_2, $\cdots$, T_m are m disjoint chains of circles at p, such that each one of these chains separates c from the origin and from a selected point on ∂D (that is, if *we consider the union of interiors of circles in any one of these chains, $Int(T_i)$, then c is contained in one connected component of $D\backslash Int(T_i)$ and the origin together with the selected point of ∂D is in the other).*

If n_1, $\cdots$, n_m are the combinatorial lengths of $T_1, T_2, \cdots, T_m$, respectively, then:

$$radius(c) \leq \frac{1}{\sqrt{n_1^{-1} + (\cdots) + n_m^{-1}}} \, .$$

Proof. Consider r_{ij} the radius of the j-th circle on the chain T_i, for $1 \leq j \leq n_i$ and $1 \leq i \leq m$. Using the Cauchy-Schwarz inequality we see that:

$$\left(\sum_{k=1}^{n_i} r_{ik}\right)^2 =< (1, \ldots, 1); (r_{i1}, \ldots, r_{in_i}) >^2 \leq (1 + \ldots + 1) \cdot (r_{1i}^2 + \ldots + r_{n_i i}^2) = n_i \cdot \sum_{k=1}^{n_i} r_{ik}^2$$

Hence, $\left(\sum_{k=1}^{n_i} r_{ik}\right)^2 \leq n_i \cdot \sum_{k=1}^{n_i} r_{ik}^2$.

Now, let ℓ_i be the the *geometric length* of the chain T_i, *i.e.*, $\ell_i = 2\sum_{k=1}^{n_i} r_{ik}$. Through the calculations above:

$$\frac{\ell_i^2}{n_i} \leq 4 \sum_{k=1}^{n_i} r_{ik}^2$$

Also, because the chains are disjoint and inside the unit disc, then the sum of their areas is less than the area of D, *i.e.*, $\sum_{i,k} \pi \cdot r_{ik}^2 \leq \pi$ and $\sum_{i,k} r_{ik}^2 \leq 1$. Therefore, using the last inequality:

$$\frac{\ell_i^2}{n_i} \leq 4$$

So:

$$\sum_{i=1}^{m} \frac{\ell_i^2}{n_i} \leq 4 \sum_{k=1}^{n_i} r_{ik}^2 \leq 4.$$

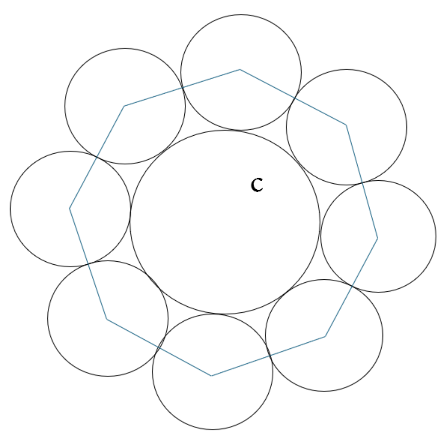
Let $l = min\{\ell_1, \cdots, \ell_m\}$, then:

$$l^2 \sum_{i=1}^{m} \frac{1}{n_i} \leq \sum_{i=1}^{m} \frac{\ell_i^2}{n_i} \leq 4$$

And:

$$l^2 \leq \frac{4}{\sum_{i=1}^{m} n_i^{-1}}$$

Besides, if a convex polygon is entirely inside another convex polygon, then the outer polygon has perimeter greater than or equal to the inner one. So, considering the inner one as the circle c and the outer polygon the one obtained by drawing the segments between centers on a chain T_i (and respective points of intersection):



We conclude that, for all $i \in \{1, \cdots, m\}$, $\sum_{k=1}^{n_i} 2\, r_{ik} \geq 2\, \pi\, radius(c) \geq 2\, radius(c)$, and $l \geq 2 \cdot radius(c)$.

Finally:

$$radius^2(c) \leq \frac{l^2}{4} \leq \frac{1}{\sum_{i=1}^{m} n_i^{-1}}$$

Which implies the end result

$$radius(c) \leq \frac{1}{\sqrt{\sum_{i=1}^{m} n_i^{-1}}}.$$

■

Theorem 29 (Hexagonal Packing Lemma)**.** *The first generation relative to a circle c in a circle packing is the union of the circles that surround c, that is, the circles tangent to c.*

The second generation relative to c is formed by the union of circles tangent to some circle in the first generation, excluding c and the circles on this first generation themselves. We may define the n-th generation recursively as one would expect.

Now, let c_0 be a circle in a circle packing P and $\epsilon > 0$.

If there is a natural number $n = n(\epsilon) \in \mathbb{N}$, such that c_0 has a (finite) induced circle packing $P_n' \subseteq P$ isomorphic to n generations of the regular hexagonal circle packing about one of its circles, then, the circles c_i immediately around c_0 satisfy the inequality:

$$\left| \frac{radius(c_i)}{radius(c_0)} - 1 \right| \leq \epsilon.$$

Remark. Hence, the deeper a circle c is in terms of the generations in an approximate regular hexagonal packing, the more similar radii of circles immediately around c are relative to radius(c).

Firstly, we'll assume a powerful lemma (whose proof involves new algebraic and analysis notions such as Schottky groups and the Lebesgue Density Theorem, hence, we shall not cover it):

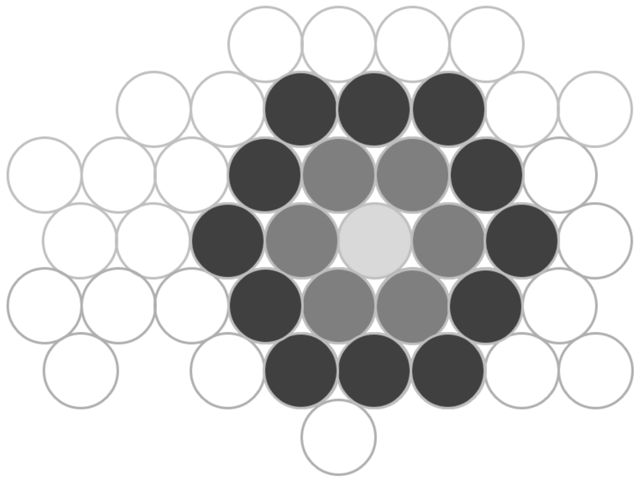


Figure 19: Generations 0, 1 and 2 relative to a circle in the regular hexagonal packing.

Lemma 14 (Rigidity of Infinite Hexagonal Packing)**.** *Any infinite circle packing with nerve isomorphic (as a graph) to the nerve of the regular hexagonal packing is exactly the regular hexagonal packing, up to compositions by reflections, translations and linear transformations.*

A detailed proof is outlined in Appendix 1 of the original article by Rodin and Sullivan, [2] or in Terence Tao's post [8]. Appealing to the geometric intuition of the reader and assuming this result, let us go forward.

Proof of Theorem 29*.* By contradiction, assume we did find a sequence of circle packings, $(\mathcal{C}_n)_n$, each isomorphic to $\mathcal{H}_n$, n generations relative to a circle $C_0 = C_{0,n} \simeq radius(c_0)\cdot S^1$, in the regular hexagonal packing which, nonetheless, satisfies the property that one of its adjacent circles $C_1 = C_{1,n}$ has radius $r_{1,n}$, going away from $radius(c_0)$ as n increases.

Applying the Ring Lemma with $C_{1,n}$ as the center circle, we have that $r_{1,n} \geq r_6 \cdot radius(c_0)$ (where r_6 is the constant from the Ring Lemma). Writing $R_n = \frac{r_{1,n}}{radius(c_0)}$, as $(R_n)_n$ is in the interval $[r_6, 1/r_6]$, the Bolzano-Weierstrass guarantees it possesses a convergent sub-sequence $(R_{n_k})_k \longrightarrow r$ for some $r \in (0, 1/r_6]$.

Generalizing this procedure, for every point z belonging to a center in a circle of $\mathcal{H}$, we know the associated circle belongs to a single generation, say $k = k(z)$. Thus, we can associate a circle $C_{z,n}$ in $\mathcal{C}_n$, for $n \geq k$. Naming $r_{z,n}$ as the radius of $C_{z,n}$, we can assume that the radii r_{z,n_k} of a sub-sequence of these circles converge to a positive number $r_{z,\infty}$ (applying the ring lemma as much as needed to get a relation between $r_{z,n}$ and $radius(c_0)$).

Furthermore, we may construct new circles $C_{z,\infty}$, for every z as before, with radius equal to $r_{z,\infty}$ and centers fabricated in order for $\bigcup_z C_{z,\infty}$ to be a circle packing (just proceed analogously with a sequence of the centers and notice that if we have a sequence of circles tangent to each other, their limits must also be intersect tangentially).

This provides a new (infinite) circle packing $\mathcal{C}_\infty = \bigcup_z C_{z,\infty}$ with nerve isomorphic to that of the (infinite) regular hexagonal packing: $\mathcal{H}$. By the previous lemma, Rigidity of Infinite Hexagonal Packing, Theorem 14, it must be an affine copy of the regular hexagonal packing which implies, first and foremost that $r_{0,\infty} = r_{1,\infty} = radius(c_0)$, contradicting the beginning statement.

■

This finishes all initial theorems we shall need for the proceeding demonstrations.

14 The Convergence of Circle Packings

We are now ready to start applying our geometric foundations to the initially mentioned scheme of the circle packings. Reminding the reader of our beginning conditions, we have a sequence $(P^n)_n = (H_n)_n$ of regular hexagonal circle packings surrounding an open, bounded, simply connected region $R \subseteq \mathbb{C}$, a sequence of simplicial mappings $(\phi_n)_n$ and their respective images, $({P'}^n)_n = (Q_n)_n$ which constitute a circle packing on the unit disc D whose border discs are internally tangent to ∂D. Our final objective is to prove that the simplicial maps $(\phi_n)_n$ converge uniformly on compact sets to a conformal Riemann mapping from R to the disc D, giving a new geometric perspective on the analytic Riemann Mapping Theorem.

To begin with, as we are about to prove convergence to the Riemann mapping, a first step should be to verify that the radii of the boundary circles of $(Q_n)_n$ will tend to 0 as n increases. Having this established, because the (nerve and, therefore) support of $(Q_n)_n$ are simply connected and its border circles are internally tangent to the border, the circle packing $(Q_n)_n$, in fact, ends up exhausting D. In computational terms, this is so that we can comfortably approximate the image of any point in R and be sure these images fill the whole disc as $n \to \infty$.

Proposition 2. *The radii of the boundary circles in $(Q_n)_n$, the circle packings on D, will tend to 0 as n goes to infinity.*

Proof. Suppose w is a boundary vertex of the nerve of $(H_n)_n$ and that $v = \phi_n^{-1}(0)$. Say that w is $k+1$ generations away from v in $(H_n)_n$, for some $k \in \mathbb{N}_0$.

Hence, there will be k mutually disjoint chains separating v from w on the region R. Writing once again n_j as the combinatorial length of the chain T_j we know that:

$$n_j \leq 6 \cdot j, \quad \forall\, 1 \leq j \leq k.$$

Since the packing $(H_n)_n$ is imposed to be hexagonal, transporting back this situation to the disc through ϕ_n, let $L_1, \cdots, L_k$ represent the corresponding chains in Q_n and c_w be the circle in Q_n corresponding to w. As before, the chains $L_1, \cdots, L_k$ will separate c_w from 0 and a point in ∂D. So, using the Length-Area lemma (13), and the fact that $n_j \leq 6 \cdot j$, comes:

$$[radius(c_w)]^2 \leq \frac{1}{\frac{1}{n_1} + (\cdots) + \frac{1}{n_k}} \leq \frac{1}{\frac{1}{6} + (\cdots) + \frac{1}{6 \cdot k}} = \frac{6}{1 + \frac{1}{2} + (\cdots) + \frac{1}{k}}$$

Notice that the denominator in the last expression is exactly the harmonic series, which diverges to ∞ as k increases, hence, $radius(c_w) \longrightarrow 0$ as $n \to \infty$. This, in fact, clarifies that the radii of the boundary circles of $(Q_n)_n$ converge to zero and that their respective supports, consequently, carriers, fill the whole unit disc D.

■

Next, we verify a previously mentioned but not confirmed statement using results from section 12: the quasi-conformality of the maps $(\phi_n)_n$.

We shall prove this recalling the first geometric lemma seen in this dissertation, the Ring Lemma and the characterization of quasi-conformal maps through the skew of equilateral triangles.

Suppose, then, that we select a point z and have an equilateral triangle T inside R, with one of the vertices the point z itself. Using the local property of quasi-conformal mappings, Theorem 26, one may suppose that there is a regular hexagonal circle packing whose nerve contains T. For notational convenience, write c_0 as the circle in this regular hexagonal packing with center z and its image: $\phi_n(T)$.

As the image of the latter regular packing is a hexagonal circle packing too (in the sense that every non-border circle is always tangent to exactly 6 other circles), we may apply the Ring lemma to the corresponding circle $\tilde{c}_0$ in D and its six surrounding circles, $\tilde{c}_i$, $1 \leq i \leq 6$. Stating, without loss of generality, that the circles with centers equal to the other two vertices of $\phi_n(T)$ are exactly $\tilde{c}_1$ and $\tilde{c}_2$ and using the Ring Lemma, we conclude:

$$radius(\tilde{c}_i) \geq r_6 \cdot radius(\tilde{c}_0), \quad \forall\, 1 \leq i \leq 2$$

Noting that we can reverse this reasoning because c_0 also surrounds $\tilde{c}_1$ and $\tilde{c}_2$, then:

$$radius(\tilde{c}_0) \geq r_6 \cdot radius(\tilde{c}_i), \quad \forall\, 1 \leq j \leq 2$$

Hence:

$$r_6 \cdot radius(\tilde{c}_0) \leq radius(\tilde{c}_i) \leq \frac{1}{r_6} \cdot radius(\tilde{c}_0), \quad \forall\, 1 \leq j \leq 2.$$

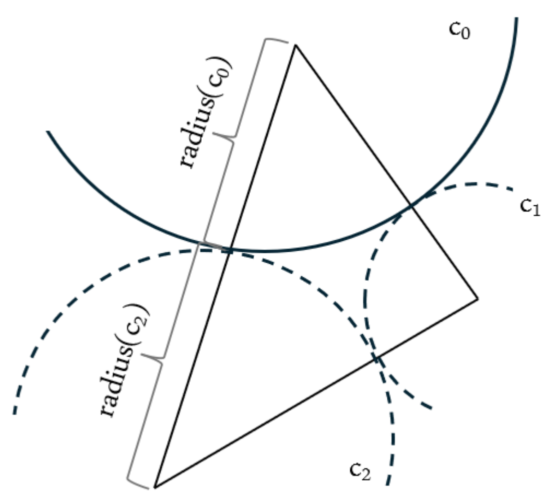


Figure 20: Calculation of skew of image triangle.

Therefore, the skew of the triangle $\phi_n(T)$ is less than or equal to $\frac{\frac{2}{r_6}\cdot radius(\tilde{c}_0)}{(2\,r_6)\cdot radius(\tilde{c}_0)} = \frac{1}{{r_6}^2}$ (see Figure 20). By Theorem 28, based on paper [1], we finally see that all maps ϕ_n are, in fact, (uniformly) quasi-conformal.

14.1 Local Uniform Convergence of The Approximating Riemann Maps

Next we will see how one can extract an uniformly convergent subsequence of $(\phi_n)_n$ on compacts through the property just proven, starting with an important new form of uniform convergence, which may not be known to some readers:

Definition 48. *A sequence $(f_n)_n$ of maps $f_n : U \longrightarrow V$ converges locally uniformly to f on U if every point of $z \in U$ has a neighborhood $W_z \subseteq U$ on which $(f_n|_{W_z})_n$ converges uniformly to $f|_{W_z}$.*

Every locally uniformly convergent sequence is compactly convergent, that is, uniformly convergent on compact subsets. An even better result allows us to state that, on (open sets of) $\mathbb{C}$, both concepts are equivalent.

Following the notes from Terence Tao in [8], one comes across the theorem below, an important step to our new objective:

Theorem 30. *Let $f_n : U \longrightarrow V_n$ be a sequence of K-quasiconformal maps for some $K \geq 1$, such that all the V_n are uniformly bounded. Then $(f_n)_n$ contains a locally uniformly convergent subsequence.*

Proof. Here is a sketch of the proof, the only missing detail I shall omit is the demonstration of the equivalence between the following new definition of quasi-conformality and the prescribed initially. All aspects are covered, once again, in [8].

Firstly, as said, it can be verified that a different and new definition of quasi-conformality may be given, the so called "modulus definition" of quasi-conformal mappings.

Consider a *topological annulus* A in the complex plane, meaning, a region bounded between two Jordan curves γ_1 and γ_2, where γ_1 is inside the interior of γ_2. In this case, we say γ_1 is the *interior curve/ boundary* of A and γ_2 is the *exterior curve/ boundary* of A.

The $(conformal)$ $modulus$ of the "$usual$" $annulus$ $A(z_0, r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$, $mod(A)$, is defined to be:

$$mod\Big(A(z_0, r, R)\Big) = log\Big(\frac{R}{r}\Big)$$

It is a known theorem that every topological annulus is conformally equivalent to an "usual" annulus: $A(0, r, R)$ (thanks to a slight modification of the Riemann Uniformization Theorem). Hence, given a topological annulus we may define its $(conformal)$ *modulus* as the modulus of its associated "usual" annulus in $\mathbb{C}$ (and this turns out to be well-defined).

This new geometric definition of K-quasiconformality ($K \geq 1$) for orientation-preserving homeomorphisms imposes these only need to satisfy:

$$\frac{1}{K} \cdot mod(A) \leq mod(f(A)) \leq K\, mod(A), \text{ for every topological annulus } A \text{ in the domain of } f.$$

Through this new perspective, it can now be shown that the $(f_n)_n$ restricted to compact subsets of U are equicontinuous making use of the succeeding argument:

Let z, w be two distinct points on a compact set, C, inside R. We can create an annulus $A = \{z : l_1 < |z - a| < l_2\}$ contained in C and such that A surrounds z and w (as in Figure 21).

As $|z - w|$ tends to 0, one can make the annulus A that contains z and w have

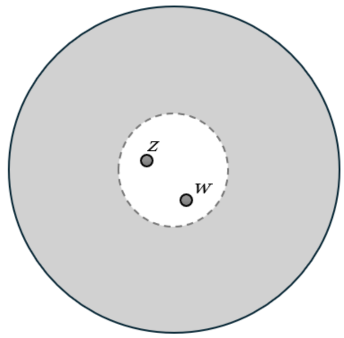


Figure 21: An annulus surrounding two points.

(conformal) modulus, $log\left(\frac{l_2}{l_1}\right)$, as large as we wish (just make l_1 sufficiently close to 0).

Also, because the ϕ_n are K-quasiconformal, $\frac{1}{K} \cdot log\left(\frac{l_2}{l_1}\right) \leq \frac{1}{K} \cdot mod(A) \leq mod(\phi_n(A))$, hence, $\phi_n(C)$ also has arbitrarily large modulus and surrounds both $\phi_n(z)$ and $\phi_n(w)$.

Nonetheless, $\phi_n(C)$ is bounded (since it is compact). Together with the arbitrarily major nature of the modulus of $\phi_n(A)$ and the fact that its exterior boundary is fixed these imply that both $\phi_n(z)$ and $\phi_n(w)$ are enclosed in a very small region and their distance apart is very small as well (independently of n). Hence, comes equicontinuity of the family $(\phi_n)_n$ on the compact C.

Through the Arzela-Ascoli Theorem, we finally conclude local uniform convergence of a subsequence in $(\phi_n)_n$. Beyond this, it was also implicitly established that any subsequence in $(\phi_n)_n$ has a convergent subsequence.

$\blacksquare$

By Theorem 30 applied to the maps $(\phi_n)_n$, there must exist a subsequence locally uniformly convergent to a function $\phi : R \longrightarrow D$.

Notice this only establishes local uniform convergence of a sub-sequence, nonetheless, in the end, we shall remedy this problem. However, from now on, us simplify the language and write the before-mentioned sub-sequence simply as $(\phi_n)_n$ (the reader should keep this simplification always in mind).

Proceeding, as each $(\phi_n)_n$ is a homeomorphism, $\phi : R \to D$ will be a continuous function. Furthermore, we may prove the same exact assertion for the inverses $(\phi_n^{-1})_n$ and conclude these also have a locally uniformly subsequence converging to a function $\psi : D \to R$ because, as the convergence is (locally) uniform, both ϕ and ψ are continuous functions. The only detail left to verify is that $\phi^{-1} \equiv \psi$.

Just notice:

$$|\phi_n \circ \phi_n^{-1}(z) - \phi \circ \psi(z)| \leq |\phi_n \circ \phi_n^{-1}(z) - \phi \circ \phi_n^{-1}(z)| + |\phi \circ \phi_n^{-1}(z) - \phi \circ \psi(z)|$$

$$\leq ||\phi_n - \phi||_\infty + |\phi \circ \phi_n^{-1}(z) - \phi \circ \psi(z)|$$

Making use of the local uniform convergence of $(\phi_n)_n$ to ϕ near $\psi(z)$ and the limit characterization of continuity in metric spaces, there exists n sufficiently large so that the last expression is less than $\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Hence, $\phi_n \circ \phi_n^{-1}$ converges pointwisely to $\phi \circ \psi$. Obviously, $\phi_n \circ \phi_n^{-1}$ also converges to the identity map, so, $\phi \circ \psi = Id$ and reversing the orders we may conclude ψ is the inverse function of ϕ.

In sum, we have come to the realization that the homeomorphisms $(\phi_n)_n$ converge locally uniformly to a homeomorphism $\phi : R \longrightarrow D$.

14.2 Approximation to The Conformal Riemann Mapping

From the previous results, we have finally concluded that the quasi-conformal homeomorphisms $(\phi_n)_n$ (better said, a sub-sequence of these) converge uniformly to a new homeomorphism $\phi : R \longrightarrow D$ on compact sets. Taking into account the statement of the

Riemann Mapping Theorem, our final purpose will be to show ϕ is quasi-conformal and, later, 1-quasiconformal, thus, a conformal mapping onto the open unit disc.

Fortunately, as ϕ is the (local) uniform limit of a sequence of quasi-conformal maps, the following proposition shall help and play an important role in this journey:

Proposition 3. *Let $K \geq 1$.*

Suppose $f_n : U \to V_n$ is a sequence of K-quasiconformal maps converging locally uniformly to a homeomorphism $f : U \to V$. Then, f is K-quasiconformal.

Proof. Since the maps $(f_n)_n$ are K-quasiconformal, the associated distortion sequence, $(H_{f_n})_n$, is bounded by K.

Claim: The dilatation sequence $(H_{f_n})_n$ converges to H_f.

Proof of claim. For n sufficiently large, we have that:

$$\Big||f_n(z) - f_n(w)| - |f(z) - f(w)|\Big| \leq \big|f_n(z) - f(z)\big| + \big|f_n(w) - f(w)\big| \leq 2 \cdot \frac{\epsilon}{2} = \epsilon$$

Therefore,

$$sup_{w \in \partial B(z,r)}\Big||f_n(z) - f_n(w)| - |f(z) - f(w)|\Big| =$$

$$= \Big|sup_{w \in \partial B(z,r)}|f_n(z) - f_n(w)| - sup_{w \in \partial B(z,r)}|f(z) - f(w)|\Big| =$$

$$= |L_{f_n}(z,r) - L_f(z,r)| \leq \epsilon$$

So $L_{f_n}(z,r) \longrightarrow L_f(z,r)$. Similarly we see that $l_{f_n}(z,r) \longrightarrow l_f(z,r)$. $\square$

Finally, since each $H_{f_n}(\,\cdot\,,r) \leq K$, $\forall\ n \in \mathbb{N}$, then $H_f(\,\cdot\,,r) \leq K$ for any small $r > 0$ and f is K-quasiconformal.

■

Thus, it follows directly from the preceding proposition and Theorem 30 on the (sub-)sequence of quasi-conformal mappings $(\phi_n)_n$ (and ϕ), that the latter is, in fact, a quasi-conformal homeomorphism.

Finally, in order to conclude our journey through the convergence of circle packings to the Riemann Mapping, the only aspect missing is:

Theorem (Convergence of Circle Packings to the Riemann Mapping)**.** *The local uniform limit of the sequence of quasi-conformal homeomorphisms $(\phi_n)_n$, in other words, the map $\phi : R \longrightarrow D$ is 1-quasiconformal, thus, conformal and a Riemann Mapping from the region R onto the open unit disc, for which $\phi(z_1) = 0$ and $\phi(z_2) \in \mathbb{R}^+$.*

Proof. Let us fix a generic point z and a compact subset S of R (as before) which contains this point, that is inside the carrier of the maps $(\phi_n)_{n \geq M}$, for some $M \in \mathbb{N}$.

Notice that, as each ϕ_n is a simplicial map, the ϕ_n are complex linear in the interior of all triangles of their respective carrier, $Carr(P^n)$.

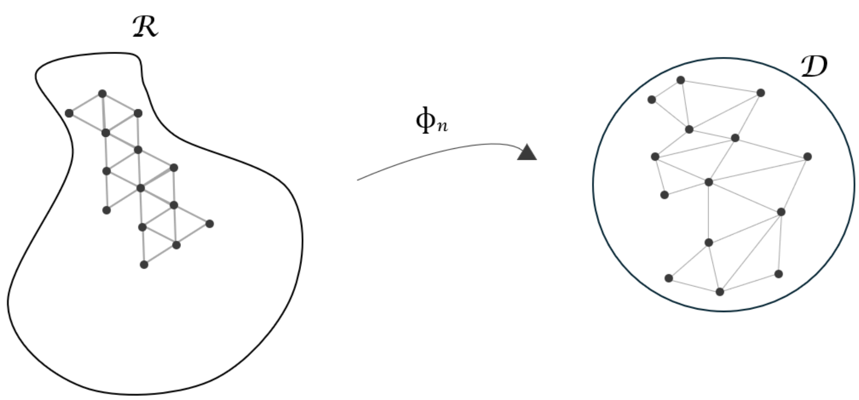


Figure 22: A figure representing the image through ϕ_n of the carrier of a regular hexagonal packing covering a compact set in R.

In particular, inside these interiors, as the maps are linear, their degree of dilation only depends on relation between the shape of a single triangle and its image.

Additionally, since the carriers of regular packings P^n become smaller and filled with more and more equilateral triangles from the regular hexagonal circle packing, by the Hexagonal Packing Lemma, 29, adjacent circles in the image by ϕ_n, ${P'}^n \subseteq D$, have radii quotient approximately equal to 1, say $1+s_n$, for all $n \in \mathbb{N}$. Once more, thanks to this same lemma, we know $(s_n)_n \longrightarrow 0$ and in the limit, as n goes to infinity, the triangles filling the whole unit disc will become arbitrarily close to equilateral (since each triangle will be surrounded by more and more generations of an hexagonal packing). Therefore, ϕ_n sends equilateral triangles to almost equilateral triangles, meaning, it is $(1+s'_n)$-quasiconformal on the interior of every triangle in the domain carrier, with s'_n another sequence convergent to zero.

From the above we, thus, conclude that the only regions where problems with high levels of dilation could originate are the edges and vertices of the carriers' triangles. Nonetheless, thanks to one of the basic results noted in the section 12, Theorem 27, we can easily circumvent this and finally conclude that each ϕ_n is everywhere $(1+s'_n)$-quasiconformal, where $(s'_n)_n \longrightarrow 0$.

Thus, as each subsequence also converges to ϕ, it must also be $(1+s'_n)$-quasiconformal, for all naturals n and we must have ϕ is 1-quasiconformal. Meaning, ϕ is a conformal mapping from R to the open unit disc.

At last, remember that in the beginning of our description, page 149, we demanded the mappings $(\phi_n)_n$ sent a fixed point $z_1^* \mapsto \phi_n(z_1^*) = 0$ and another $z_2^* \mapsto \phi_n(z_2^*) > 0$. Consequently, $\phi(z_1^*) = 0$ and $\phi(z_2^*) > 0$. As a result, the uniqueness of the Riemann mapping from R to the disc, assured by Schwarz Lemma, guarantees that the sub-subsequences of ϕ_n all converge to the the same unique Riemann mapping. This implies, due to the Urysohn sub-subsequence principle, that the the original sequence $(\phi_n)_n$, in fact, also converges locally uniformly on R to the (unique) conformal mapping $\phi : R \longrightarrow D$ such that

$\phi(z_1^*) = 0$ and $\phi(z_2^*) > 0$.

Our ultimate purpose has now been corroborated and the end result establishes we can, as matter of fact, approximate all Riemann mappings in this amazing, innovative and beautiful geometric vision.

$\blacksquare$

15 Conclusion

In sum, throughout this dissertation we have introduced the notion of Riemann surfaces, diving deeper into new concepts of Complex Analysis such as the Dirichlet Problem, the study of holomorphic, meromorphic, harmonic and sub-harmonic functions on Riemann surfaces together with many important properties intrinsic to them, such as: the Harnack's Principle, the $\mathcal{L}^2$ space of 1-forms and the Mean-Value property, among many others. Finishing the first section with the proof of the Riemann Uniformization Theorem which ensures the classification of all simply connected Riemann surfaces in just three categories: $\mathbb{C}, \mathbb{CP}^1$ or the unit disc $\mathbb{D}$. This is a powerful result which, even though only applies to simply connected surfaces, can (with a bit more work) allow us to characterize all Riemann Surfaces as a quotient of one of the three previously mentioned manifolds by a group acting freely and properly discontinuous, providing, thus, complete knowledge of all complex structured 2-manifolds.

Furthermore, an interesting geometric approach to the Riemann Mapping Theorem was supplied, treated and proven with the use of circle packings, tying together the illustrative world of graphs with the analytic perspective of the Riemann mappings. The method of approximation in question using affine extensions between carriers of circle packings, quasi-conformal functions and the theorem presented in the article [1] which turns many of the original steps in the original Rodin and Sullivan demonstration simpler.

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